A semi-Lagrangian scheme for the game p-Laplacian

Stefano FINZI VITA
(Sapienza Università di Roma)

(joint work with M. Falcone [Roma I], T. Giorgi and R.G. Smits [New Mexico])

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Outline

1. Introduction: two player games and game p-Laplacians
2. p-averages and discrete p-Laplacians
3. The approximation scheme and its convergence
4. Numerical implementation
5. Numerical tests

p-Laplacians and applications

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- Granular materials: models for growing sandpiles can be derived as limits of fast/slow diffusion problems in terms of p-Laplacians
- Stochastic game theory: random-turn games can be used in economical and political modeling, real world conflicts where opposing agents continually seek to improve their positions through incremental *tugs*, the move sets are player-symmetric but independent of what the others do.

Game p-Laplacians: definition

$$\Delta_p^G u := \frac{1}{p} |\nabla u|^{2-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = \frac{1}{p} |\nabla u|^{2-p} \Delta_p u \quad (1 \leq p < \infty)$$

$$\Delta_\infty^G u := |\nabla u|^{-2} \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = |\nabla u|^{-2} \Delta_\infty u$$

(in red are the usual variational p-Laplacians)
These operators are *degenerate elliptic* and *singular* for any $p \neq 2$.
Remark that $\Delta_2^G u = \frac{1}{2} \Delta_2 u$.
By expanding the derivatives:

$$\Delta_p^G u = \frac{1}{p} \Delta_2 u + \frac{p-2}{p} |\nabla u|^{-2} \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad (1)$$

we see that for $p \to \infty$ formally $\Delta_p^G u$ converges to $\Delta_\infty^G u$.

Remarks

- When u is twice differentiable and $\nabla u \neq 0$,

$$\Delta_\infty^G u =< D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} >,$$

where $D^2 u$ denotes the Hessian matrix, that is the second derivative of u in the gradient direction.

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- It can be easily seen that

$$\Delta_1^G u = |\nabla u| \Delta_1 u = \Delta_2 u - \Delta_\infty^G u,$$

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- Combining previous relation with (1), we get the interesting characterization:

$$\Delta_p^G = \frac{1}{p} \Delta_1^G + \frac{1}{q} \Delta_\infty^G, \tag{2}$$

  (q conjugate exponent of p), that is any game p-Laplacian can be thought as the convex combination of the two limiting cases.

- Since Δ_p^G and Δ_p differ only by the factor $p|\nabla u|^{p-2}$, when the homogeneous Dirichlet problem is treated, the distinction between the two operators is irrelevant (same p-harmonic functions in classical, weak and viscosity sense).

- But we are interested to study the general non homogeneous Dirichlet problem
$$-\Delta_p^G u = f \quad \text{in } \Omega, \qquad u = F \quad \text{on } \partial\Omega \tag{3}$$
which has no variational sense, but a natural game-theoretic interpretation.

- In general, solutions are not twice differentiable: $u(x,y) = |x|^{4/3} - |y|^{4/3}$ is an example of an ∞-harmonic function in the square $\Omega = (-1,1)^2$ which is not C^2. [Aronsson, '67]

- The usual definition of viscosity solution for nonlinear partial differential equations has to be extended in this case to take care of degeneracy and singularity.

Game interpretation

- In the classical case ($p = 2$) the homogeneous Dirichlet problem ($\Delta u = 0$ in Ω, $u = F$ on $\partial\Omega$) can be solved by starting a Brownian motion B at x, running it until the hitting time τ of the boundary, and taking $u(x) = E_x[F(B(\tau))]$.
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  - $p = \infty$: tug-of-war game
  - $1 < p < \infty$: tug-of-war with noise

Motion by mean curvature ($p = 1$) [Kohn-Serfaty, '06]

The equation for motion of level sets by mean curvature can be interpreted in terms of the following deterministic two player game. Let $\Omega \in \mathbb{R}^2$, $x \in \Omega$.

- Player 1 wants to reach the boundary, player 2 tries to obstruct him.
- At each step player 1 choses a direction $v \in \mathbb{R}^2$, player 2 replaces v with $\pm v$ (i.e. stand or reverse v), then player 1 moves from x to $x + \sqrt{2}\varepsilon bv$.
- The value function of the game is the minimum exit time, given by $u_\varepsilon(x) = \varepsilon^2 k$ if player 1 needs k steps to exit Ω starting from x and following an optimal strategy.
- For $\varepsilon \to 0$ (the continuum limit) u_ε converges to the solution of

$$-\Delta_1 u = 1 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Random turn games

Two-player zero-sum games

- X set of states, $Y \subset X$ nonempty set of terminal states (target)
- $F : Y \to \mathbb{R}$ terminal payoff function,
  $f : X \setminus Y \to \mathbb{R}$ running payoff function
- E_1, E_2 transition graphs with vertex set X

The game:

- A token is initially placed at $x_0 \in X \setminus Y$, the initial state
- At the k-th step a fair coin is tossed and the player who wins may move the token to any x_k s.t. (x_{k-1}, x_k) is a directed edge in the transition graph
- The game ends the first time $x_k \in Y$, with player 1's payoff

$$F(x_k) + \sum_{i=0}^{k-1} f(x_i)$$

- Player 1 seeks to maximize this payoff, Player 2 to minimize it

The tug-of-war game ($p = \infty$) [Peres-Schramm-Sheffield-Wilson '08]

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- each player tries to "tug" the token to his own target and away from his opponent one; the game ends when a target is reached
- the value of the game when the game starts at x is given by
  $u_1(x) = \sup_{S_1} \inf_{S_2} \hat{F}(S_1, S_2)$ for player 1,
  $u_2(x) = \inf_{S_2} \sup_{S_1} \hat{F}(S_1, S_2)$ for player 2,
  where S_1, S_2 are the strategies for the two players, and $\hat{F}$ denotes the expected total payoff at the termination of the game

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  where S_1, S_2 are the strategies for the two players, and $\hat{F}$ denotes the expected total payoff at the termination of the game
- the game has a value u when $u_1(x) = u_2(x) = u(x)$

The tug-of-war game on a metric spaces

- (X, d) metric space, $Y \subset X$
- E_ε edge-set s.t. $x \sim y$ iff $d(x, y) < \varepsilon$
- $u^\varepsilon(x)$ value of the ε-t.o.w. game with terminal payoff F and running payoff $\varepsilon^2 f$ which starts at $x = x_0 \in X \setminus Y$:
  - at step k a coin is tossed and the winner choses x_k s.t. $d(x_k, x_{k-1}) < \varepsilon$
  - game ends when $x_k \in Y$, with payoff $F(x_k) + \varepsilon^2 \sum_{i=0}^{k-1} f(x_i)$
- $u(x) = \lim_{\varepsilon \to 0} u^\varepsilon(x)$ (if $\exists$) is the *continuum value* of the t.o.w. game

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $F : \partial\Omega \to \mathbb{R}$ uniformly continuous, $f : \Omega \to \mathbb{R}$ either zero or strictly positive uniformly continuous.
Then there exists a unique $u : \overline{\Omega} \to \mathbb{R}$ continuous viscosity solution of

$$-\Delta_\infty^G u = 2f \quad \text{in } \Omega, \qquad u = F \quad \text{on } \partial\Omega \tag{4}$$

which is the continuum value of the tug-of-war on (Ω, d, F, f).

The tug-of-war with noise $(1 < p < \infty)$ [Peres-Sheffield '08]

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- at each step k the player who wins the toss chooses a vector v_k s.t. $|v_k| \leq \varepsilon$ and set $x_k = x_{k-1} + v_k + z_k$, where z_k is a random noise vector orthogonal to v_k with lenght $\sqrt{\frac{1}{p-1}}|v_k|$

  ($p \to \infty$: $|z_k| \to 0$, tug-of-war; $p = 2$: $|z_k| = |v_k|$, random walk)

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- the game ends when the boundary is reached at some point y and player 1's payoff is set to $F(y)$ (Player 1 tries to reach the boundary, player 2 tries to prevent that)
- $u^\varepsilon(x)$ is the value of the game for player 1 starting from point $x \in \Omega$

The Tug-of-war game with noise

TUG-OF-WAR WITH NOISE AND THE p-LAPLACIAN 93

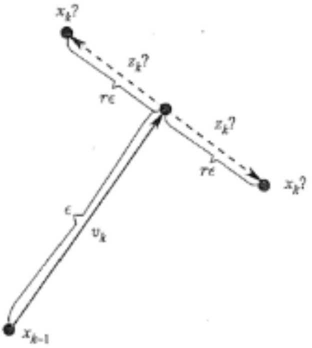

Figure 1. A move of tug-of-war with noise in dimension 2 for the noise distribution μ given by $\mu\{(0,r)\} = \mu\{(0,-r)\} = 1/2$. The player who wins the coin toss adds a vector v_k of length at most ϵ to the game position x_{k-1}, and then a random noise vector z_k with law μ_{v_k} (and magnitude $r|v_k|$) is added to produce x_k. In the figure, $|v_k| = \epsilon$.

The Tug-of-war game with noise

- If Ω is *game-regular* (*), as $\varepsilon \to 0$ the functions u^ε converge uniformly to the unique p-harmonic extension u of F

  (*) For any $y \in \partial\Omega$, if the game starts near y, player 1 has a strategy for making the game terminate near y with high probability.
  Sufficient conditions:
  - $p > d$ (in $\mathbb{R}^d$)
  - $\partial\Omega$ satisfies the cone property
  - Ω simply connected (in $\mathbb{R}^2$)

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- If we add a running payoff of size $\varepsilon^2 f(x_k)$ at the k-th step, u will be the solution of problem

$$-\Delta_p^G u = \frac{2f}{q} \quad \text{in } \Omega, \qquad u = F \quad \text{on } \partial\Omega \tag{5}$$

Viscosity solutions in $\mathbb{R}^2$

Definition

Given $1 < p \leq \infty$, an upper semi-continuous function [respectively, lower semi-continuous] $u : \Omega \to \mathbb{R}$ is a viscosity subsolution [supersolution] of

$$-\Delta_p^G u(x) = f(x) \text{ in } \Omega, \tag{6}$$

if for any $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum [local minimum] at $x \in \Omega$, we have

(i) $-\Delta_p^G \phi(x) \leq f(x) \quad [-\Delta_p^G \phi(x) \geq f(x)] \quad$ if $\nabla\phi(x) \neq 0$;

(ii) $-\Delta_2^G \phi(x) \leq f(x) \quad [-\Delta_2^G \phi(x) \geq f(x)] \quad$ whenever $\nabla\phi(x) = 0$.

A function u is a viscosity solution of (6) if u is a viscosity subsolution and supersolution according to (i) and (ii) .

Numerical schemes for the p-Laplacian problems

For the variational p-Laplacian several approximation schemes have been proposed, using

- Finite element methods (Barrett-Liu '94)
- Finite difference methods for degenerate second order operators (Crandall-Lions '96, Oberman '04)
- Finite volumes methods (Andreyanov-Boyer-Hubert '06)

Here we present a FD approach which is strictly connected with the game theory interpretation of the problems, and which then applies to both homogeneous (where variational and game Laplacian problems have the same solutions) and non-homogeneous cases.

The key tool is the notion of p-average.

Definition

Given a finite set of real numbers, $S = \{s_1, s_2, \ldots.., s_m\}$, we denote by $A_p(S)$ the p-average of its elements, that is $A_p(S)$ is such that

$$\sum_{j=1}^{m} |s_j - A_p(S)|^p = \min_{c\in\mathbb{R}} \sum_{j=1}^{m} |s_j - c|^p \quad \text{if} \quad 1 < p < \infty, \tag{7}$$

$$A_\infty(S) = \frac{1}{2}\left[\max_{s_j\in S} s_j + \min_{s_j\in S} s_j\right],$$

$$A_1(S) = median\ (S).$$

- By convexity $A_p(S)$ is uniquely defined for $1 < p \leq \infty$.
- If $s_1 \leq s_2 \leq .. \leq s_m$: $A_1(S) = \begin{cases} s_{k+1} & \text{if } m = 2k+1, \\ (s_k + s_{k+1})/2 & \text{if } m = 2k. \end{cases}$
- An easy calculation shows that $A_2(S)$ is the usual arithmetic mean

Properties of the $p - average$

- For any $k \in \mathbb{R}$: $A_p(S+k) = A_p(S) + k$
- $\min_{j=1..m} s_j \leq A_p(S) \leq \max_{j=1..m} s_j$
- Let $S = \{s_1, s_2, ..., s_m\}$ and $T = \{t_1, t_2, ..., t_m\}$ be two finite sets of real numbers having the same number m of elements, and let $1 \leq p \leq \infty$ be fixed. If $t_j \leq s_j$, $\forall j = 1..m$, then $A_p(T) \leq A_p(S)$.
- Let S and T be two finite sets of real numbers having the same number of elements, and let $1 \leq p \leq \infty$ be fixed. Assume that $S = \{s_1, s_2, ..., s_m\}$ and $T = \{t_1, t_2, ..., t_m\}$ verify $t_j = s_j + \delta_j$, for every $j = 1..m$, where $|\delta_j| < \delta$ for some $\delta > 0$, then

$$A_p(S) - \delta \leq A_p(T) \leq A_p(S) + \delta.$$

p-averages and approximation schemes for the p-Laplacian

For $p = 2$ we can rewrite the classical 5-points finite difference formula for the two dimensional Laplacian in terms of 2-average:

$$\begin{aligned} \Delta_2^G\, u(x_1, x_2) \quad &\approx \frac{1}{2\,h^2}\,[u(x_1 + h, x_2) + u(x_1, x_2 + h) \\ &+ u(x_1 - h, x_2) + u(x_1, x_2 - h) - 4\,u(x_1, x_2)] = \\ &= \frac{2}{h^2}\,[A_2(C_h(\mathbf{x}, u)) - \; u(\mathbf{x})]\,, \end{aligned}$$

where $C_h(\mathbf{x}, u)$ is the set of the four values of u in the adjacent nodes, that is

$$C_h(\mathbf{x}, u) = \{u(x_1 + h, x_2), u(x_1, x_2 + h), u(x_1 - h, x_2), u(x_1, x_2 - h)\}.$$

We could pick as $C_h(\mathbf{x}, u)$ even a larger set of values of u on the sphere of radius h, but since the Laplacian is a linear operator, this would not increase the accuracy.

p-averages and approximation schemes for the p-Laplacian

For $p = \infty$ ([Oberman, '04a]) :

$$\Delta_\infty^G u(\mathbf{x}) \approx \frac{2}{h^2} \left[A_\infty(C_h(\mathbf{x}, u)) - u(\mathbf{x}) \right],$$

where now $C_h(\mathbf{x}, u)$ is a discrete set of values of u on the $B(\mathbf{x}, h)$, and the distribution and number of points on the sphere influences the accuracy of the approximation.

For $p = 1$ a similar approach has been proposed by [Oberman, '04b]

This suggests the following generalization to the game p-Laplacian $\forall p$:

$$\Delta_p^G u(\mathbf{x}) \approx \frac{2}{h^2} \left[A_p(C_h(\mathbf{x}, u)) - u(\mathbf{x}) \right], \tag{8}$$

where $C_h(\mathbf{x}, u)$ would be a suitable discrete set of values of u on $B(\mathbf{x}, h)$.

The semi-Lagrangian scheme

We are then lead to the following approximation scheme for problem (3):

$$S(\rho, \mathbf{x}, u(\mathbf{x}), u) = 0 \text{ in } \overline{\Omega}, \tag{9}$$

where $\rho := (h, \Delta\theta)$ (with h spatial step and $\Delta\theta$ angular resolution), and $S : [0,1) \times (0, \pi/2] \times \overline{\Omega} \times \mathbb{R} \times L^\infty(\overline{\Omega}) \longrightarrow \mathbb{R}$ defined as

$$S(\rho, \mathbf{x}, u(\mathbf{x}), u) = \begin{cases} -\frac{2}{\alpha^2 h^2} \left[A_p(C_h^{\Delta\theta}(\mathbf{x}, u; \alpha)) - u(\mathbf{x})\right] - f(\mathbf{x}) & \text{in } \Omega, \\ u(\mathbf{x}) - F(\mathbf{x}) & \text{on } \partial\Omega. \end{cases}$$

If $d_\Omega < \infty$ denotes the diameter of Ω, $\alpha = \alpha(\mathbf{x})$ is a dilation parameter such that $0 < \alpha^* < \alpha(\mathbf{x}) \leq dist(\mathbf{x}, \partial\Omega) < d_\Omega$.
$C_h^{\Delta\theta}(\mathbf{x}, u; \alpha)$ is a suitably chosen discrete set of values of u, taken on the sphere $B(\mathbf{x}, h\alpha)$, associated to the angular resolution $\Delta\theta$.

Convergence results

- The scheme is monotone:
  if $u, v \in L^\infty(\overline{\Omega})$, $u(\mathbf{x}) \geq v(\mathbf{x})$ in $\overline{\Omega}$, then for all $p \geq 1$, $\rho \in [0,1) \times (0, \pi/2]$, $\mathbf{x} \in \overline{\Omega}$, $t \in \mathbb{R}$ it holds
  $$S(\rho, \mathbf{x}, t, u) \leq S(\rho, \mathbf{x}, t, v).$$

Convergence results

- The scheme is monotone:
  if $u, v \in L^\infty(\overline{\Omega})$, $u(\mathbf{x}) \geq v(\mathbf{x})$ in $\overline{\Omega}$, then for all $p \geq 1$, $\rho \in [0,1) \times (0, \pi/2]$, $\mathbf{x} \in \overline{\Omega}$, $t \in \mathbb{R}$ it holds
  $$S(\rho, \mathbf{x}, t, u) \leq S(\rho, \mathbf{x}, t, v).$$
- For any $p \geq 2$ the scheme is consistent:
  for all $\mathbf{x} \in \Omega$ and $\phi \in C^\infty(\overline{\Omega})$, we have that

$$\lim_{\rho \to 0} \frac{2}{\alpha^2 h^2} \left[A_p(C_h^{\Delta\theta}(\mathbf{x}, \phi; \alpha)) - \phi(\mathbf{x}) \right] = \begin{cases} \Delta_p^G \phi(\mathbf{x}) & \text{if } \nabla \phi(\mathbf{x}) \neq 0, \\ \Delta_2^G \phi(\mathbf{x}) & \text{if } \nabla \phi(\mathbf{x}) = 0. \end{cases}$$

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- The scheme is stable (when $f \equiv 0$):
  $\forall h > 0, \Delta\theta > 0$, there exists a solution $u_\rho \in L^\infty(\overline{\Omega})$ of $S(\rho, x, t, u_\rho) = 0$ such that $\|u_\rho\|_\infty \leq \|F\|_\infty$.

Convergence follows by a result of Barles-Souganidis, '91

"Any monotone, stable and consistent approximation scheme to fully nonlinear second order elliptic or parabolic, possibly degenerate, PDE converges to the correct (viscosity) solution provided that there exists a comparison principle for the limiting equation."

Assume $f \equiv 0$, then there exists a unique bounded viscosity solution for problem (2), for which a comparison principle holds.
Since the scheme is monotone, stable and consistent (for $p \geq 2$), then it is *convergent*:

The solution u_ρ of (9) converges as $\rho \to 0$ to the unique viscosity solution of (3).

Numerical implementation

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- Numerical tests show that they converge to the right viscosity solution of the stationary problem

Space discretization and interpolation

- $h > 0$ space discretization step ($\{\mathbf{x}_j\}_{j=1}^N$ nodes on a h-uniform grid)
- $\Delta\theta = \frac{\pi}{2m}$ angular discretization step ($\rightarrow 4m$ points on the disk)
- $C_h^{\Delta\theta}(\mathbf{x}, u; \alpha)) = \{u(\mathbf{y}^i), i = 0, 1, .., 4m-1\}$ set for the p-average, with $\mathbf{y}^i = \mathbf{x} + h\alpha r_i, \quad r_i = (\cos i\Delta\theta, \sin i\Delta\theta)$

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- $\mathbf{y}^i$ not in general grid points $\rightarrow$ bilinear interpolation
- $\widehat{C}_h^{\Delta\theta}(\mathbf{x}_j, u; \alpha_j) = \{I[u](\mathbf{y}_j^i), i = 0, 1, .., 4m-1\}$
  - $I[u](\mathbf{y}) = ay_1y_2 + by_1 + cy_2 + d = \sum_{k=1}^4 u(\mathbf{x}_k)\lambda_k(\mathbf{y})$
  - $\mathbf{x}_k$ four vertices of the cell where $\mathbf{y}$ is
  - $\lambda_k(\mathbf{y})$ given functions dependings on the coordinates of $\mathbf{x}_k$
  - α_j may vary from point to point ($\rightarrow$ *multi-level circle stencil*) $\alpha_j(\mathbf{x}_j) = \beta \min(s, d_j/h) \quad (s \in \mathbb{N},\ 0 < \beta \leq 1,\ d_j = dist(\mathbf{x}_j, \partial\Omega))$

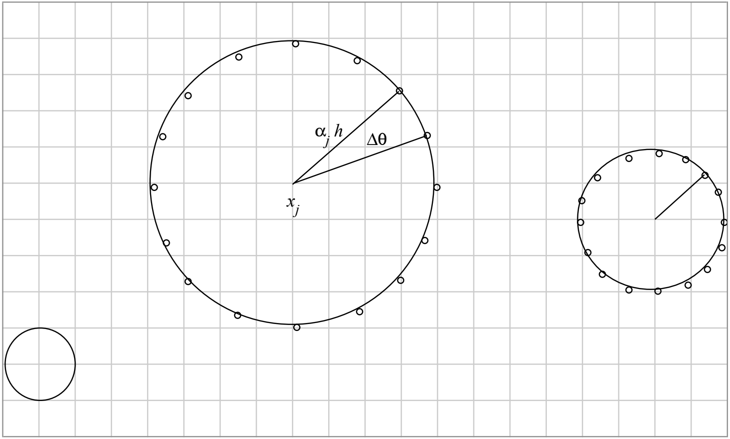

Figure: The 4-level circle stencil.

Remarks

- The choice of bilinear interpolation is essential to preserve the monotonicity of the scheme which is needed for the convergence proof. Quadratic interpolation has not this property.
  Bilinear interpolation satisfies:
  - $\min_D u(x) \leq I[u](x) \leq \max_D u(x)$
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- With our choise of $\Delta\theta$ the number of equally distributed points on the sphere of radius $h\alpha_j$ is a multiple of 4. This means that if $\mathbf{r}$ is an admissible direction, so is its opposite $-\mathbf{r}$, as well as its orthogonal and its reflections with respect of each of the axes.

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- The tests show that the multi-level circle strategy is able in general to speed up the convergence (reducing the number of iterations).

The practical scheme

On the previous grid, given an initial condition $\mathbf{u}^0 \in \mathbb{R}^N$, we implement the explicit time marching scheme (Δt is the time discretization step):

$$\mathbf{u}^{n+1} = T_\rho(\mathbf{u}^n) := \mathbf{u}^n - \Delta t \; S(\rho, \mathbf{y}, \mathbf{u}^n(\mathbf{y}), \mathbf{u}^n),$$

which is consistent, because the scheme S is consistent in the stationary case. Taking care of the interpolation, we get

$$u_j^{n+1} = \begin{cases} u_j^n + \dfrac{2\,\Delta t}{\alpha_j^2\, h^2}\left[A_p(\widehat{C}_h^{\Delta\theta}(\mathbf{x}_j, \mathbf{u}^n; \alpha_j)) - \; u_j^n\right] + \Delta t \; f(\mathbf{x}_j) & \mathbf{x}_j \in \Omega, \\ F(\mathbf{x}_j) & \mathbf{x}_j \in \partial\Omega; \end{cases}$$

until a stopping criterion is satisfied ($\|\mathbf{u}^{n+1} - \mathbf{u}^n\|_\infty \le \varepsilon$).

A simpler variant

If we choose parameters s.t. $\frac{2\Delta t}{\alpha_j^2 h^2} = 1$ the scheme simplifies as

$$u_j^{n+1} = \begin{cases} A_p(\widehat{C}_h^{\Delta\theta}(\mathbf{x}_j, \mathbf{u}^n; \alpha_j)) + \Delta t\, f(\mathbf{x}_j) & \text{if } \mathbf{x}_j \in \Omega, \\ \\ F(\mathbf{x}_j) & \text{if } \mathbf{x}_j \in \partial\Omega; \end{cases}$$

Remark. The p-averages can be computed by any standard method for minimization of convex function. We used the *Newton Bracketing method*, but this part can be optimized to improve the speed of calculations.

The Newton-Bracketing method

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function such that its minimum value f_{min} is attained, and is contained in the bracket $[L, U = f(x)]$ for some point x such that $f'(x) \neq 0$. Then the iterative method generates a sequence of nested brackets shrinking to a point:

1. Stopping rule: If $U - L < \varepsilon$ stop with x as a solution.
2. Select a value $M := \alpha U + (1 - \alpha)L$ for some $0 < \alpha < 1$.
3. Do one Newton iteration $x_+ = x - \frac{f(x)-M}{f'(x)}$.
4. Case 1: If $f(x_+) < f(x)$ then update U : $U_+ := f(x_+)$ and leave $L_+ := L$. Go to 1.
5. Case 2: If $f(x_+) \geq f(x)$ then update L : $L_+ := M$ and leave $U_+ := U$, $x_+ := x$. Go to 1.

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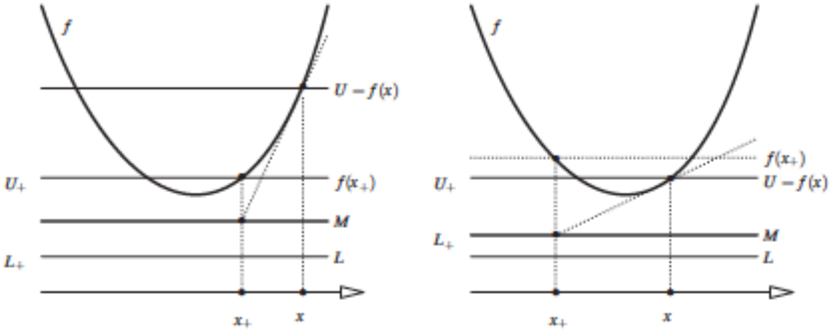


(a) Case 1: $f(x_+) < f(x)$ (b) Case 2: $f(x_+) \geq f(x)$

Fig. 4.1 Illustration of the 2 cases of the NB method

Convergence of the iterations

- Let $\dfrac{2\,\Delta t}{\alpha_*^2\, h^2} \leq 1$. Assume $f \equiv 0$, then for $n \geq 1$ it holds

$$\sup_{j=1..N} |u_j^n| \leq \sup_{j=1..N} |u_j^{n-1}| \leq \sup_{j=1..N} |u_j^0| \quad (\text{stability})$$

- Let $\dfrac{2\,\Delta t}{\alpha_*^2\, h^2} \leq 1$ and $\mathbf{u}^0$ given by

$$u_j^0 = \begin{cases} \min\limits_{\partial\Omega} F & \text{if } \mathbf{x}_j \in \Omega, \\ F(\mathbf{x}_j) & \text{if } \mathbf{x}_j \in \partial\Omega. \end{cases}$$

  Then for $n \geq 1$ the iterations generated by the scheme verify

$$u_j^n \geq u_j^{n-1} \text{ for any } j = 1..N, \text{ and } n \geq 1 \quad (\text{pointwise monotonicity})$$

- Then for $f \equiv 0$ and an appropriately chosen initial condition the scheme is pointwise convergent.

Example 1: Aronsson function

$$\Omega = (-1,1)^2, \quad p = \infty, \quad f \equiv 0, \quad F(x,y) = |x|^{4/3} - |y|^{4/3}$$

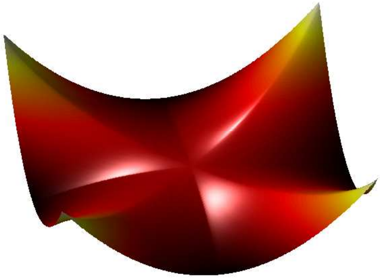
Then $u = |x|^{4/3} - |y|^{4/3}$ is the exact solution of the problem (an example of an absolute minimizer which is not twice differentiable). The scheme gives the following results (N^2= nodes, d= directions) when a 2-level iterations is used (with $\beta = 0.99$):

d	$N = 41$	$N = 81$	$N = 161$	$N = 241$	$N = 401$
4	0.1105 (250)	0.0765 (448)	0.0373 (584)	0.0225 (589)	0.0122 (621)
8	0.0274 (80)	0.0182 (161)	0.0084 (214)	0.0069 (190)	0.0048 (188)
16	0.0084 (54)	0.0070 (75)	0.0043 (105)	0.0033 (108)	0.0023 (112)
24	0.0088 (57)	0.0081 (73)	0.0050 (91)	0.0035 (103)	0.0024 (107)

Table: L^∞- errors and iterations (in parentheses) for test on Aronsson function

Example 2

$$\Omega = (-1,1)^2, \quad p = \infty, \quad f = 0, \quad F(x,y) = |x|^2\,|y|^2$$



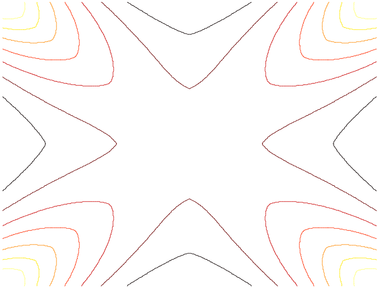


Figure: $N = 401^2$, $d = 24$, 2-lev ($\beta = 0.99$).

Example 3

$$\Omega = (-1,1)^2, \quad p = \infty, \quad f = 0, \quad F(x,y) = x^3 - 3xy^2$$

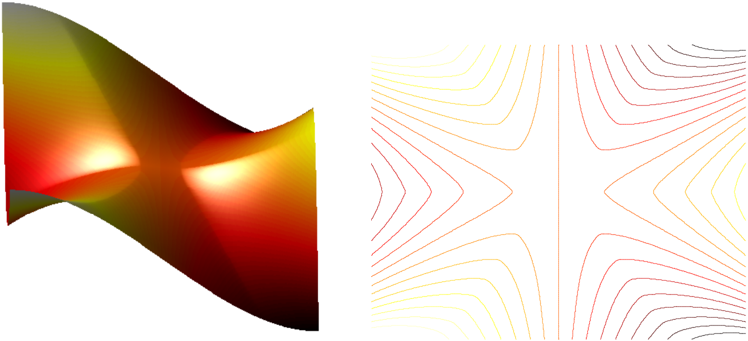
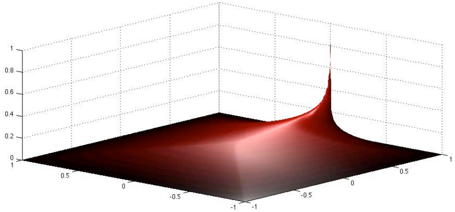


Figure: $N = 401^2$, $d = 24$, 2-lev ($\beta = 0.99$).

Example 4

$\Omega = (-1,1)^2$, $p = \infty$, $f = 0$, F characteristic function of point $(1,0)$



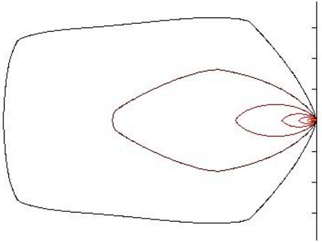


Figure: $N = 401^2$, $d = 24$, 4-lev ($\beta = 0.99$).

Example 5

$$\Omega = (-1,1)^2, \quad p \geq 2, \quad f = 1, \quad F(x,y) = (1 - x^2 - y^2)/2$$

The exact solution is known: $u(x,y) = (1 - x^2 - y^2)/2$, the function which solves the problem with $F = 0$ on the unit sphere.

N (lev)	$d = 16$	$d = 24$
21 (2)	0.0634 (163)	0.0617 (180)
21 (4)	0.0241 (50)	0.0192 (107)
41 (4)	0.0201 (213)	0.0191 (163)

Table: L^∞-errors and iterations for $p = 5$, $\beta = 0.9$.

N (lev)	$d = 16$	$d = 24$
21 (2)	0.0590 (249)	0.0563 (248)
21 (4)	0.0211 (80)	0.0185 (77)
41 (4)	0.0192 (272)	0.0156 (272)

Table: L^∞-errors and iterations for $p = \infty$, $\beta = 0.9$.

Example 6

$$\Omega = (-2,2) \times (-1,1), \quad p = \infty, \quad f \equiv 1, \quad F \equiv 0$$

The exact solution is known only in part of the domain:
$u(x,y) = (1-y^2)/2$ when $|x| \leq 1$

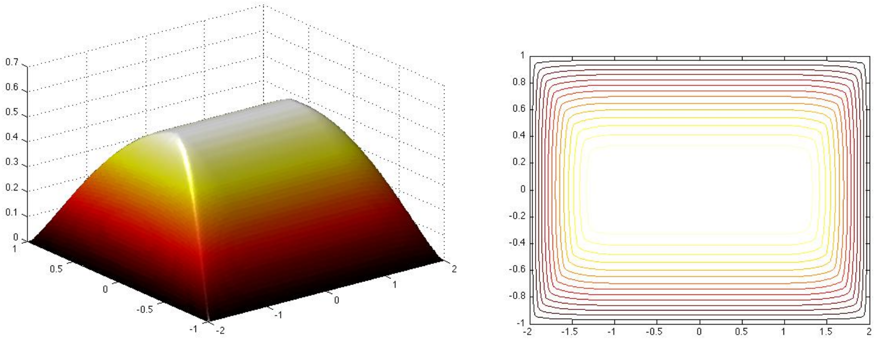


Figure: Surface and contour plots for $N = 201 \times 101$, $d = 16$, 4-lev ($\beta = 0.8$).

Thank you