An optimal transportation problem with import/export taxes on the boundary

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Schedule

1. Statement of the problem
2. Mass Transport Theory
3. The mass transport problem with import/export taxes: Duality
4. Evans-Gangbo Performance
5. Optimal Transport Plans
6. Limited importation/exportation

Statement of the problem

Business

A business man (BM) produces some product in some factories in Ω:

$$f_+ \quad \text{gives the amount of product and its location.}$$

There are some consumers of the product in Ω:

$$f_- \quad \text{gives its distribution.}$$

BM wants:

to transport all the mass f_+ to f_- (to satisfy the consumers) or to the boundary (export).

Costs and constrains

The BM pays:

the transport costs (given by the Euclidean distance)   PLUS

when a unit of mass is left on a point $y \in \partial\Omega$: $T_e(y)$, the export taxes.

BM must satisfy all the demand of the consumers. So that, he has to import product, if necessary, from the exterior, paying:

$T_i(x)$, import taxes, for each unit of mass that enters Ω from $x \in \partial\Omega$.

Some freedom

BM has the freedom to chose to export or import mass

provided

he transports all the mass in f_+ and covers all the mass of f_-.

MAIN GOAL

The main goal here is to minimize the total cost of this operation, that is given by the transport cost PLUS export/import taxes.

Given the set of transport plans

$$\mathcal{A}(f_+, f_-) := \left\{ \mu \in \mathcal{M}^+(\overline{\Omega} \times \overline{\Omega}) : \begin{array}{l} \pi_1 \# \mu \llcorner \Omega = f_+ \mathcal{L}^N \llcorner \Omega, \\ \pi_2 \# \mu \llcorner \Omega = f_- \mathcal{L}^N \llcorner \Omega \end{array} \right\},$$

the goal is to obtain

$$\min_{\mu \in \mathcal{A}(f_+, f_-)} \int_{\overline{\Omega} \times \overline{\Omega}} |x - y| \, d\mu + \int_{\partial\Omega} T_i d(\pi_1 \# \mu) + \int_{\partial\Omega} T_e d(\pi_2 \# \mu)$$

$\pi_i : \mathbb{R}^N \times \mathbb{R}^N$ are the projections: $\pi_1(x, y) := x$, $\pi_2(x, y) := y$.
Given a Radon measure γ in $X \times X$, $\pi_1 \# \gamma$ is the marginal $proj_x(\gamma)$, and $\pi_2 \# \gamma$ is the marginal $proj_y(\gamma)$.

Equilibrium of masses

The transport must verify the natural equilibrium of masses, nevertheless it is not necessary to impose

$$\int_\Omega f_+(x)\,dx = \int_\Omega f_-(y)\,dy\,.$$

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In this transport problem two masses appear on $\partial\Omega$, that are unknowns, the ones that encode the mass exported and the mass imported.

Taking into account this masses we must have the equilibrium condition.

Conditions on the domains

Ω is assumed to be convex:

we prevent the fact that the boundary of Ω is crossed when the mass is transported inside Ω.

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On the support of the $f_\pm$ we assume that

$$\mathrm{supp}(f_+) \cap \mathrm{supp}(f_-) = \emptyset.$$

Mass Transport Theory

Monge problem

Given two measures $\mu, \nu \in \mathcal{M}(X)$ (say $X \subset \mathbb{R}^N$) satisfying the mass balance condition

$$\int_X d\mu = \int_X d\nu,$$

Monge's Problem consists in solving

$$\inf_{T\#\mu=\nu} \int_X |x - T(x)| d\mu(x),$$

$T : X \to \mathbb{R}^N$ are Borel functions pushing forward μ to ν: $T\#\mu = \nu$,

$$T\#\mu(B) = \mu(T^{-1}(B)).$$

A minimizer T^* is called an *optimal transport map* of μ to ν.

In the case that μ and ν represent the distribution for production and consumption of some commodity, the problem is to decide which producer should supply each consumer minimizing the total transport cost.

Relaxed Problem: Monge-Kantorovich Problem

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Fix two measures μ, ν satisfying the mass balance condition. Let $\Pi(\mu, \nu)$ the set of transport plans γ between μ and ν:

$$\mathrm{proj}_x(\gamma) = \mu, \ \mathrm{proj}_y(\gamma) = \nu.$$

The problem of finding a measure $\gamma^* \in \Pi(\mu, \nu)$ which minimizes the cost functional

$$\mathcal{K}(\gamma) := \int_{X \times X} |x - y| \, d\gamma(x, y),$$

in the set $\Pi(\mu, \nu)$, is well-posed.

A minimizer γ^* is called an *optimal transport plan* between μ and ν.

Linearity makes the Monge-Kantorovich problem simpler than Monge original problem: a continuity-compactness argument guarantees the existence of an optimal transport plan.

Dual Problem: Kantorovich Duality

Linear minimization problems admit a dual formulation:

For $(\varphi, \psi) \in L^1(d\mu) \times L^1(d\nu)$, define

$$J(\varphi, \psi) := \int_X \varphi \, d\mu + \int_X \psi \, d\nu,$$

and let Φ be the set of all measurable functions

$$(\varphi, \psi) \in L^1(d\mu) \times L^1(d\nu) \quad \text{satisfying}$$

$$\varphi(x) + \psi(y) \leq |x - y| \qquad \text{for } \mu \times \nu - \text{almost all } (x, y) \in X \times X.$$

Then

$$\inf_{\gamma \in \Pi(\mu,\nu)} \mathcal{K}(\gamma) = \sup_{(\varphi,\psi) \in \Phi} J(\varphi, \psi).$$

And... Kantorovich-Rubinstein Theorem

Theorem (Kantorovich-Rubinstein)

$$\min_{\gamma\in\Pi(\mu,\nu)} \mathcal{K}(\gamma) = \sup_{u\in K_1(X)} \int_X u\, d(\mu-\nu),$$

where

$$K_1(X) := \{u : X \to \mathbb{R} \ : \ |u(x)-u(y)| \leq |x-y| \ \ \forall\, x,y \in X\}$$

is the set of 1-Lipschitz functions in X*.*

The maximizers u^* of the right hand side are called *Kantorovich (transport) potentials*.

Evans and Gangbo Approach

For $\mu = f_+ \mathcal{L}^N$ and $\nu = f_- \mathcal{L}^N$, f_+ and f_- adequate Lebesgue integrable functions, Evans and Gangbo find a Kantorovich potential as a limit, as $p \to \infty$, of solutions to

$$\begin{cases} -\Delta_p u_p = f_+ - f_- & \text{in } \ B(0,R), \\ u_p = 0 & \text{on } \ \partial B(0,R). \end{cases}$$

Theorem

$$u_p \rightrightarrows u^* \in K_1(\Omega) \qquad \textit{as } p \to \infty.$$

$$\int_\Omega u^*(x)(f_+(x) - f_-(x))\,dx = \max_{u \in K_1(\Omega)} \int_\Omega u(x)(f_+(x) - f_-(x))\,dx.$$

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Characterization of the Kantorovich potential: There exists $0 \leq a \in L^\infty(\Omega)$ *such that*

$$f_+ - f_- = -\mathrm{div}(a \nabla u^*) \quad \textit{in } \; \mathcal{D}'(\Omega).$$

Furthermore $|\nabla u^*| = 1$ *a.e. in the set* $\{a > 0\}$.

Evans and Gangbo Approach

Evans and Gangbo used this PDE to find a proof of the existence of an optimal transport map for the classical Monge problem, different to the first one given by Sudakov in 1979 by means of probability methods.

The mass transport problem with import/export taxes

First approach: Duality

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* to pick it up at $x \in \partial\Omega$, $T_i(x)$ (paying the taxes by himself),

* to leave the product at the consumer's location $f_-(y)$, $y \in \Omega$, he will charge him $\psi(y)$, and

* for leaving it at $y \in \partial\Omega$, $T_e(y)$ (paying again the taxes himself).

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* for leaving it at $y \in \partial\Omega$, $T_e(y)$ (paying again the taxes himself).
* He will do all that job in such a way that:

$$\varphi(x) + \psi(y) \leq |x - y|$$

$$-T_i \leq \varphi \quad \text{and} \quad -T_e \leq \psi \quad \text{on } \partial\Omega.$$

(assuming negative payments if necessary).

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(assuming negative payments if necessary).
Observe that we need to impose the following condition:

$$-T_i(x) - T_e(y) \leq |x - y|, \; \forall\, x, y \in \partial\Omega,$$

(there is not benefit of importing mass from x and exporting it to y).
For $x = y \in \partial\Omega$: $T_i(x) + T_e(x) \geq 0$ means that in the same point the sum of exportation and importation taxes is non negative.

Clever Fellow's Goal

Consider

$$J(\varphi,\psi) := \int_\Omega \varphi(x) f_+(x)\,dx + \int_\Omega \psi(y) f_-(y)\,dy,$$

defined for continuous functions, and let

$$\mathcal{B}(T_i, T_e) := \left\{ \begin{array}{r} (\varphi,\psi) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : \varphi(x) + \psi(y) \le |x-y|, \\ -T_i \le \varphi, \ -T_e \le \psi \text{ on } \partial\Omega \end{array} \right\}.$$

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Now, given $(\varphi,\psi) \in \mathcal{B}(T_i,T_e)$ and $\mu \in \mathcal{A}(f_+,f_-)$:

$$\begin{aligned} J(\varphi,\psi) &= \int_\Omega \varphi(x) f_+(x)\,dx + \int_\Omega \psi(y) f_-(y)\,dy \\ &= \int_{\overline{\Omega}} \varphi(x) d\pi_1\#\mu - \int_{\partial\Omega} \varphi d\pi_1\#\mu + \int_{\overline{\Omega}} \psi(y) d\pi_2\#\mu - \int_{\partial\Omega} \psi d\pi_2\#\mu \end{aligned}$$

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$$\begin{aligned} J(\varphi,\psi) &= \int_\Omega \varphi(x) f_+(x)\, dx + \int_\Omega \psi(y) f_-(y)\, dy \\ &= \int_{\overline{\Omega}} \varphi(x) d\pi_1\#\mu - \int_{\partial\Omega} \varphi d\pi_1\#\mu + \int_{\overline{\Omega}} \psi(y) d\pi_2\#\mu - \int_{\partial\Omega} \psi d\pi_2\#\mu \\ &\le \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu + \int_{\partial\Omega} T_i d(\pi_1\#\mu) + \int_{\partial\Omega} T_e d(\pi_2\#\mu). \end{aligned}$$

Acceptance of the deal

Therefore,

$$\sup_{(\varphi,\psi)\in\mathcal{B}(T_i,T_e)} J(\varphi,\psi)$$
$$\leq \inf_{\mu\in\mathcal{A}(f_+,f_-)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu + \int_{\partial\Omega} T_i d(\pi_1\#\mu) + \int_{\partial\Omega} T_e d(\pi_2\#\mu).$$

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This inequality will imply that the business man accept the offer.

NO GAP

But, in fact, there is no gap between both costs:

Theorem

Assume that T_i and T_e satisfy

$$- T_i(x) - T_e(y) < |x-y| \quad \forall\, x, y \in \partial\Omega.$$

Then,

$$\sup_{(\varphi,\psi)\in\mathcal{B}(T_i,T_e)} J(\varphi,\psi)$$
$$= \min_{\mu\in\mathcal{A}(f_+,f_-)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu + \int_{\partial\Omega} T_i d\pi_1\#\mu + \int_{\partial\Omega} T_e d\pi_2\#\mu\,.$$

Proof based on Fenchel-Rocafellar's duality Theorem.

UNKNOWNS

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there are masses, the ones that appear on the boundary, that are unknown variables.

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there are masses, the ones that appear on the boundary, that are unknown variables.

The above result is an abstract result and does not give explicitly the measures involved in the problem. But ...

Evans-Gangbo Performance

Kantorovich potentials and optimal export/import masses

Let $f \in L^\infty(\Omega)$ and $N < p < +\infty$.
Let $g_i \in C(\partial\Omega)$, $g_1 \leq g_2$ on $\partial\Omega$.
Set $W^{1,p}_{g_1,g_2}(\Omega) = \{u \in W^{1,p}(\Omega) : g_1 \leq u \leq g_2 \text{ on } \partial\Omega\}$
and consider the energy functional

$$\Psi_p(u) := \int_\Omega \frac{|\nabla u(x)|^p}{p}\, dx - \int_\Omega f(x)u(x)\, dx.$$

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$W^{1,p}_{g_1,g_2}(\Omega)$ is a closed convex subset of $W^{1,p}(\Omega)$ and Ψ_p is convex, l.s.c. and coercive. Then

$$\min_{u \in W^{1,p}_{g_1,g_2}(\Omega)} \Psi_p(u)$$

has a minimizer u_p in $W^{1,p}_{g_1,g_2}(\Omega)$, which is a least energy solution of the obstacle problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ g_1 \leq u \leq g_2 & \text{on } \partial\Omega. \end{cases}$$

Theorem

Assume g_1, g_2 *satisfy*

$$g_1(x) - g_2(y) \leq |x - y| \qquad \forall\, x, y \in \partial\Omega.$$

Then, up to subsequence,

$$u_p \rightrightarrows u_\infty \quad \textit{uniformly,}$$

and u_∞ *is a maximizer of the variational problem*

$$\max\left\{ \int_\Omega w(x) f(x)\, dx : \, w \in W^{1,\infty}_{g_1,g_2}(\Omega),\ \|\nabla w\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

Theorem

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$$\max \left\{ \int_\Omega w(x) f(x)\, dx \colon w \in W^{1,\infty}_{g_1,g_2}(\Omega),\ \|\nabla w\|_{L^\infty(\Omega)} \le 1 \right\}.$$

A natural question:

Let f_+ and f_- be the positive and negative part of f.
Can u_∞ be interpreted as a kind of Kantorovich potential for some transport problem involving f_+ and f_- ?

DUALITY

Theorem

If

$$g_1(x) - g_2(x) \leq |x-y| \quad x, y \in \partial\Omega,$$

then

$$\int_\Omega u_\infty(x)(f_+(x) - f_-(x))dx = \sup_{(\varphi,\psi)\in\mathcal{B}(-g_1,g_2)} J(\varphi,\psi)$$

$$= \min_{\mu\in\mathcal{A}(f_+,f_-)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu - \int_{\partial\Omega} g_1 d\pi_1\#\mu + \int_{\partial\Omega} g_2 d\pi_2\#\mu\,,$$

where u_∞ is the maximizer given in the above Theorem.

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where u_∞ is the maximizer given in the above Theorem.

Observe that setting $T_i = -g_1$ and $T_e = g_2$ we are dealing with our transport problem.

Key result

Theorem

Assume $g_1(x) - g_2(y) < |x - y| \; \forall\, x, y \in \partial\Omega$.
Let u_p, u_∞ *the functions obtained above. Then, up to a subsequence,*

$$\mathcal{X}_p := |Du_p|^{p-2} Du_p \to \mathcal{X} \quad \text{weakly* in the sense of measures,}$$

$$-div(\mathcal{X}) = f \quad \text{in the sense of distributions in } \Omega,$$

Moreover, the distributions $\mathcal{X}_p \cdot \eta$ *defined as*

$$\langle \mathcal{X}_p \cdot \eta, \varphi \rangle := \int_\Omega \mathcal{X}_p \cdot \nabla\varphi - \int_\Omega f\varphi \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^N),$$

are Radon measures supported on $\partial\Omega$*, that, up to a subsequence,*

$$\mathcal{X}_p \cdot \eta \to \mathcal{V} \quad \text{weakly* in the sense of measures;}$$

and ...

Key result

u_∞ is a Kantorovich potential for the classical transport problem of the measure

$$\hat{f}_+ = f_+ \mathcal{L}^N \llcorner \Omega + \mathcal{V}^+ \quad \textit{to} \quad \hat{f}_- = f_- \mathcal{L}^N \llcorner \Omega + \mathcal{V}^-.$$

Moreover,

$$u_\infty = g_1 \quad \textit{on} \quad \textit{supp}(\mathcal{V}^+),$$
$$u_\infty = g_2 \quad \textit{on} \quad \textit{supp}(\mathcal{V}^-).$$

Key result

This result provides a proof for the previous duality Theorem:

$$\int_{\overline{\Omega}} u_\infty d(\hat{f}_+ - \hat{f}_-) = \min_{\nu \in \Pi(\hat{f}_+, \hat{f}_-)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_0,$$

for some $\nu_0 \in \Pi(\hat{f}_+, \hat{f}_-)$. Then, since $\pi_1 \# \nu_0 \llcorner \partial\Omega = \mathcal{V}^+$ and $\pi_2 \# \nu_0 \llcorner \partial\Omega = \mathcal{V}^-$,

$$\int_{\Omega} u_\infty (f_+ - f_-) = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_0 - \int_{\partial\Omega} g_1 \, d\pi_1 \# \nu_0 + \int_{\partial\Omega} g_2 \, d\pi_2 \# \nu_0 .$$

Moreover, we have that

$\mathcal{V}^+$ and $\mathcal{V}^-$ are import and export masses in our original problem.

Proof

Let u_p be a minimizer of

$$\Psi_p(u) := \int_\Omega \frac{|\nabla u(x)|^p}{p}\, dx - \int_\Omega f(x)u(x)\, dx.$$

Then,

$$\begin{cases} -\Delta_p u_p = f & \text{in } \Omega, \\ g_1 \le u_p \le g_2 & \text{on } \partial\Omega. \end{cases}$$

Proof

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Then,

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Therefore for $\mathcal{X}_p := |Du_p|^{p-2} Du_p$ we have that

$$-\mathrm{div}(\mathcal{X}_p) = f \quad \text{in the sense of distributions,}$$

and that the distribution $\mathcal{X}_p \cdot \eta$, defined as

$$\langle \mathcal{X}_p \cdot \eta, \varphi \rangle := \int_\Omega \mathcal{X}_p \cdot \nabla \varphi - \int_\Omega f \varphi \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^N),$$

is supported on $\partial\Omega$.

We proof that, in fact, $\mathcal{X}_p \cdot \eta$ is the sum of a nonnegative and a nonpositive distribution, so it is a measure, and moreover that

$$\mathrm{supp}((\mathcal{X}_p \cdot \eta)^+) \subset \{x \in \partial\Omega : u_p(x) = g_1(x)\},$$

and

$$\mathrm{supp}((\mathcal{X}_p \cdot \eta)^-) \subset \{x \in \partial\Omega : u_p(x) = g_2(x)\}.$$

In addition, we have that

$$\int_\Omega \mathcal{X}_p \cdot \nabla\varphi = \int_\Omega f\varphi + \int_{\partial\Omega} \varphi \, d\,(\mathcal{X}_p \cdot \eta) \quad \forall \varphi \in W^{1,p}(\Omega).$$

Boundedness of u_p, $\mathcal{X}_p$ and $\mathcal{X}_p \cdot \eta$

By using adequate test functions we get that, for $p \geq N+1$,

$$\int_\Omega |\nabla u_p|^p \leq C,$$

$$\int_{\partial\Omega} d(\mathcal{X}_p \cdot \eta)^{\pm} \leq C,$$

C constant not depending on p.

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C constant not depending on p.

Therefore, there exists a sequence $p_i \to \infty$ such that

$$u_{p_i} \rightrightarrows u_\infty \quad \text{uniformly in } \overline{\Omega}, \quad \text{with } \|\nabla u_\infty\|_\infty \leq 1,$$

$$\mathcal{X}_{p_i} \rightharpoonup \mathcal{X} \quad \text{weakly}^* \text{ as measures in } \Omega,$$

and

$$\mathcal{X}_{p_i} \cdot \eta \rightharpoonup \mathcal{V} \quad \text{weakly}^* \text{ as measures on } \partial\Omega.$$

Then

$$\int_{\Omega} \nabla\varphi \, d\mathcal{X} = \int_{\Omega} f\varphi \, dx + \int_{\partial\Omega} \varphi d\mathcal{V} \quad \forall \varphi \in C^1(\overline{\Omega}).$$

That is, formally,

$$\begin{cases} -\mathrm{div}(\mathcal{X}) = f & \text{in } \Omega \\ \\ \mathcal{X} \cdot \eta = \mathcal{V} & \text{on } \partial\Omega. \end{cases}$$

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$$\begin{cases} -\mathrm{div}(\mathcal{X}) = f & \text{in } \Omega \\ \\ \mathcal{X}\cdot\eta = \mathcal{V} & \text{on } \partial\Omega. \end{cases}$$

But also we prove that

$$\int_\Omega f u_\infty + \int_{\partial\Omega} u_\infty \, d\mathcal{V} = \int_\Omega d|\mathcal{X}|.$$

And this allows to prove that for any 1–Lipschitz continuous function w,

$$\int_\Omega u_\infty f \, dx + \int_{\partial\Omega} u_\infty d\mathcal{V} \geq \int_\Omega wf \, dx + \int_{\partial\Omega} wd\mathcal{V}.$$

Giving $\varphi \in C^1(\overline{\Omega})$ with $\|\nabla\varphi\|_\infty \leq 1$,

$$\int_\Omega u_\infty f\,dx + \int_{\partial\Omega} u_\infty d\mathcal{V} = \int_\Omega d|\mathcal{X}| \geq \int_\Omega \frac{\mathcal{X}}{|\mathcal{X}|}\cdot\nabla\varphi\,d|\mathcal{X}|$$
$$= \int_\Omega \nabla\varphi\,d\mathcal{X} = \int_\Omega \varphi f\,dx + \int_{\partial\Omega}\varphi d\mathcal{V}.$$

Then, by approximation, given a Lipschitz continuous function w with $\|\nabla w\|_\infty \leq 1$, we obtain

$$\int_\Omega u_\infty f\,dx + \int_{\partial\Omega} u_\infty d\mathcal{V} \geq \int_\Omega wf\,dx + \int_{\partial\Omega} wd\mathcal{V}.$$

Conclusion

Therefore u_∞ is a Kantorovich potential for the classical transport problem associated to the measures

$$f_+ \, d\mathcal{L}^N \llcorner \Omega + \mathcal{V}^+ \quad \text{and} \quad f_- \, d\mathcal{L}^N \llcorner \Omega + \mathcal{V}^-$$

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(the mass of both measures is the same).

Moreover,

$\mathcal{V}^+$ and $\mathcal{V}^-$ are import and export masses in our original problem.

Optimal Transport Plans

Detailing the transport

Once export/import masses on the boundary are fixed, $\mathcal{V}^+$ and $\mathcal{V}^-$, take an optimal transport plan μ for

$$f_+\mathcal{L}^N \llcorner \Omega + \mathcal{V}^+ \quad \text{and} \quad f_-\mathcal{L}^N \llcorner \Omega + \mathcal{V}^-,$$

(remark that necessarily $\mu(\partial\Omega \times \partial\Omega) = 0$).

The part of f_+ exported is

$$\tilde{f}_+ = \pi_1\#(\mu \llcorner \overline{\Omega} \times \partial\Omega),$$

and the part of f_- imported is

$$\tilde{f}_- = \pi_2\#(\mu \llcorner \partial\Omega \times \overline{\Omega}).$$

An optimal transport plan via transport maps. Kantorovich potential.

$\exists\, t_2 : \mathrm{supp}(\tilde{f}_+) \to \partial\Omega$ an optimal map pushing $\tilde{f}_+$ forward $\mathcal{V}^+$.

$\exists\, t_1 : \mathrm{supp}(\tilde{f}_-) \to \partial\Omega$ an optimal map pushing $\tilde{f}_-$ forward $\mathcal{V}^-$.

$\exists\, t_0 : \mathrm{supp}(f_+) \to \Omega$ an optimal map pushing $f_+ - \tilde{f}_+$ forward to $f_- - \tilde{f}_-$.

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Then,

$$\mu^*(x,y) = \tilde{f}_+(x)\delta_{y=t_2(x)} + (f_+(x) - \tilde{f}_+(x))\delta_{y=t_0(x)} + \tilde{f}_-(y)\delta_{x=t_1(y)}$$

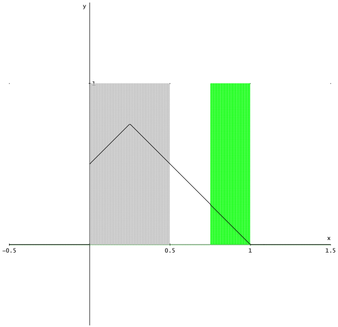
is an optimal transport plan for our problem.

u_∞ *is a Kantorovich potential for each of these transports problems.*

$$u_\infty(x) = \min_{y\in\partial\Omega}(g_2(y) + |x - y|) \quad \textit{for a.e. } x \in supp(\tilde{f}_+),$$

$$u_\infty(x) = \max_{y\in\partial\Omega}(g_1(y) - |x - y|) \quad \textit{for a.e. } x \in supp(\tilde{f}_-).$$

$T_i = 0, \;\; T_e = \frac{1}{2}$



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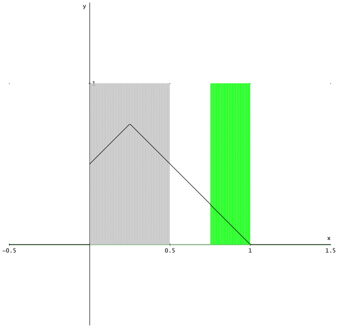
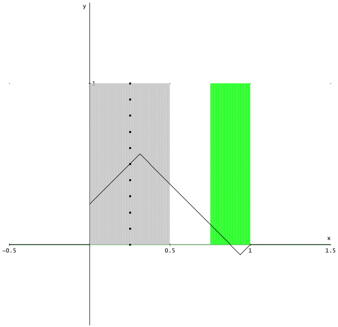


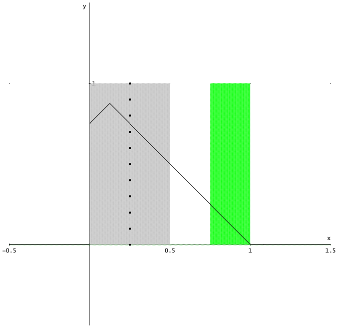
Importing mass from 1 is free of taxes, nevertheless, doing this would imply to export more mass to 0 and export taxes make this expensive.

$T_i = 0, \;\; T_e = \frac{1}{4}$



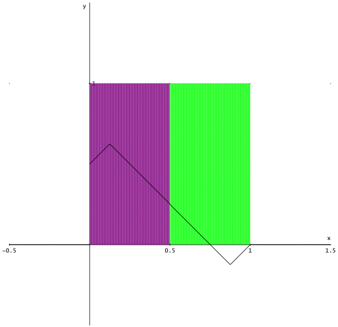
Decreasing the export taxes, more mass is exported to 0 and some mass is imported from 1.

$T_i = 0, \ T_e(0) = \frac{3}{4}, \ T_e(1) = 0$



Increasing export taxes in 0, all the other null, implies that the optimal transport plan exports to 1.

$T_i = 0, \;\; T_e = \frac{1}{2}$



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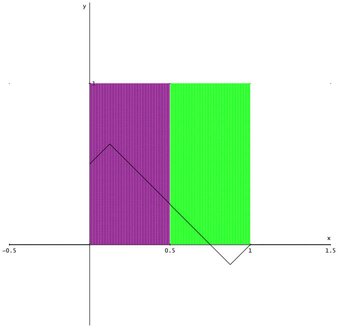

Observe that this transport problem would have coincided with the classical one if we had put $T_e = 1$.

Limited importation/exportation

More realistic

We have been using $\partial\Omega$ as an infinite reserve/repository.

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We can impose the restriction of not exceeding the punctual quantity of $M_e(x)$ when we export some mass through $x \in \partial\Omega$ and also a punctual limitation of $M_i(x)$ for importing from $x \in \partial\Omega$.
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A natural constraint must be verified:

$$-\int_{\partial\Omega} M_e \leq -\int_{\Omega} (f_+ - f_-) \leq \int_{\partial\Omega} M_i,$$

that is, the interplay with the boundary is possible.

Hence the mass transport problem is possible, and the problem becomes, as before, to minimize the cost.

Especial situations

1. When moreover $T_i = T_e = 0$: limited importation/exportation but without taxes.

2. When $M_i(x) = 0$ or $M_e(x) = 0$ on certain zones of the boundary: it is not allowed importation or exportation in those zones.

2′. $M_i(x) = M_e(x) = 0$ on the whole boundary ($\int_\Omega f_+ = \int_\Omega f_-$) we recover the classical Monge-Kantorovich mass transport problem.

New goal

Given now

$$\mathcal{A}_\ell(f_+, f_-) := \left\{ \begin{array}{ll} \mu \in \mathcal{M}^+(\overline{\Omega} \times \overline{\Omega}) : & \pi_1 \# \mu \llcorner \Omega = f_+ \mathcal{L}^N \llcorner \Omega, \\ & \pi_2 \# \mu \llcorner \Omega = f_- \mathcal{L}^N \llcorner \Omega, \\ & \pi_1 \# \mu \llcorner \partial\Omega \leq M_i, \\ & \pi_2 \# \mu \llcorner \Omega \leq M_e \end{array} \right\},$$

we want to obtain

$$\min_{\mu \in \mathcal{A}_\ell(f^+, f^-)} \int_{\overline{\Omega} \times \overline{\Omega}} |x - y| \, d\mu + \int_{\partial\Omega} T_i d(\pi_1 \# \mu) + \int_{\partial\Omega} T_e d(\pi_2 \# \mu).$$

Conditions on T_i and T_e: $T_i + T_e \geq 0$ on $\partial\Omega$.

The Clever Fellow again

New deal:
* to pick up the product $f_+(x)$, $x \in \Omega$, he will charge him $\varphi(x)$,
* to pick it up at $x \in \partial\Omega$, $T_i(x)$ (paying the taxes by himself)
* to leave the product at the consumer's location $f_-(y)$, $y \in \Omega$, he will charge him $\psi(y)$, and
* for leaving it at $y \in \partial\Omega$, $T_e(y)$ (paying again the taxes himself).

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* he will do all this in such a way that:

$$\varphi(x) + \psi(y) \leq |x - y|$$

$$\cancel{-T_i \leq \varphi \quad \text{and} \quad -T_e \leq \psi \quad \text{on } \partial\Omega,}$$

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(now:)
* he will pay a compensation when $\varphi < -T_i$ or $\psi < -T_e$ of:

$$\int_{\partial\Omega} M_i(-\varphi - T_i)^+ + \int_{\partial\Omega} M_e(-\psi - T_e)^+.$$

The aim of the fellow now is to obtain

$$\sup_{\varphi(x)+\psi(y)\leq|x-y|} \int_\Omega \varphi f_+ + \int_\Omega \psi f_- - \int_{\partial\Omega} M_i(-\varphi - T_i)^+ - \int_{\partial\Omega} M_e(-\psi - T_e)^+.$$

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And again

There is no gap between both costs.

New energy functional

The energy functional to be considered now is

$$\Psi_\ell(u) := \int_\Omega \frac{|\nabla u(x)|^p}{p}\,dx - \int_\Omega f(x)u(x)\,dx + \int_{\partial\Omega} j(x,u(x)),$$

$$j(x,r) = \begin{cases} M_i(x)(-T_i(x) - r) & \text{if } r < -T_i(x), \\ 0 & \text{if } -T_i(x) \le r \le T_e(x), \\ M_e(x)(r - T_e(x)) & \text{if } r > T_e(x). \end{cases}$$

For the minimizers of

$$\min_{u \in W^{1,p}(\Omega)} \Psi_\ell(u),$$

we prove the following result:

Theorem

Up to subsequence,

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There exist $\mathcal{X}_p \cdot \eta \in L^\infty(\partial\Omega)$, $-\mathcal{X}_p \cdot \eta \in \partial j(x, u_p)$ *a.e.* $x \in \partial\Omega$ *such that*

$$\int_{\partial\Omega} \mathcal{X}_p \cdot \eta\, \varphi = \int_\Omega |Du_p|^{p-2} Du_p \cdot \nabla\varphi - \int_\Omega f\varphi \quad \textit{for all } \varphi \in W^{1,p}(\Omega),$$

so u_p *are solutions of the nonlinear boundary problem*

$$\begin{cases} -\Delta_p u = f & \textit{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \eta + \partial j(\cdot, u(\cdot)) \ni 0 & \textit{on } \partial\Omega. \end{cases}$$

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$\mathcal{X}_p \cdot \eta \to \mathcal{V}$ *weakly$*$ in* $L^\infty(\partial\Omega)$ *with*

$$\mathcal{V}^+ \le M_i, \ \mathcal{V}^- \le M_e.$$

Theorem

$$\sup_{w\in W^{1,\infty}(\Omega),\ \|\nabla w\|_{L^\infty(\Omega)}\leq 1} \int_\Omega w(x)f(x)\,dx - \int_{\partial\Omega} j(x,w(x))$$

$$= \int_\Omega u_\infty(x)(f^+(x) - f^-(x))dx - \int_{\partial\Omega} j(x,u_\infty(x))$$

$$= \sup_{\varphi(x)+\psi(y)\leq|x-y|} \int_\Omega \varphi f_+ + \int_\Omega \psi f_- - \int_{\partial\Omega} M_i(-\varphi - T_i)^+ - \int_{\partial\Omega} M_e(-\psi - T_e)^+$$

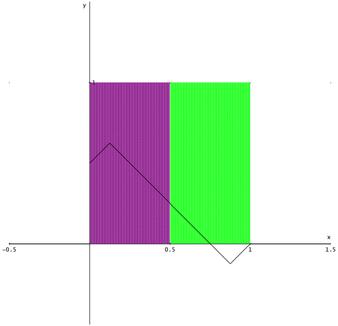
$$= \min_{\mu\in\mathcal{A}_\ell(f^+,f^-)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\mu + \int_{\partial\Omega} T_i d\pi_1\#\mu + \int_{\partial\Omega} T_e d\pi_2\#\mu\,.$$

u_∞ is a Kantorovich potential for the classical transport problem for the measures

$$f^+\mathcal{L}^N \llcorner \Omega + \mathcal{V}^+ \, d\mathcal{H}^{N-1} \llcorner \partial\Omega \quad \text{and} \quad f^-\mathcal{L}^N \llcorner \Omega + \mathcal{V}^- \, d\mathcal{H}^{N-1} \llcorner \partial\Omega.$$

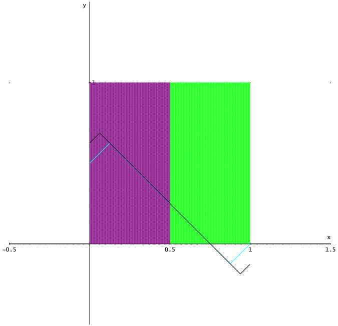
$\mathcal{V}^+ \leq M_i$ and $\mathcal{V}^- \leq M_e$ are the import and export masses.

$T_i = 0,\ T_e = \frac{1}{2},\ M_i = M_e = \infty$



Cost: $56/16^2$.

$T_i = 0, \ T_e = \frac{1}{2}, \ M_i = M_e = \frac{1}{16}$



Cost: $58/16^2$.

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