Sensitivity analysis for a class of semi-coercive variational inequalities using recession tools.

Samir ADLY
Université de Limoges
France

Workshop International
sur les mathématiques et l'environnement
Essaouira, 23-24 novembre 2012.

References

The talk is based on the following papers :

- K. ADDI, S. ADLY, D. GOELEVEN, H. SAOUD, *A sensitivity analysis of a class of semi-coercive variational inequalities using recession tools*, JOGO 2007.
- K. ADDI, S. ADLY, B. BROGLIATO, D. GOELEVEN, *A method using the approach of Moreau and Panagiotopoulos for the mathematical formulation of non-regular circuits in electronics*, Nonlinear Analysis, Hybrid Systems 2007.
- S. ADLY, E. ERNST, M. THÉRA, *Stability of Non-coercive Variational Inequalities*, Communications in Contemporary Mathematics Vol. 4, 1, 145-160 (2002).

Outline

1. Non-coercive Variational inequalities : a state of art
2. Stability of semi-coercive variational inequalities
3. Ideal diode model : a complementarity formulation
4. Set-Valued Ampere-Volt characteristics in electronics : the convex case
5. Some applications in electronics and mechanics

Outline

Position of the problem

Let us consider the problem :

$$V.I.(A, f, \Phi, K) \begin{cases} \text{Find } u \in K \text{ such that} \\ \langle Au - f, v - u \rangle + \Phi(v) - \Phi(u) \geq 0, \\ \forall v \in K \end{cases}$$

- X is a reflexive Banach space ,
- $K \subset X$ is a closed convex subset (nonemty),
- $A : X \to X^*$ is an operator,
- $f \in X^*$,
- $\Phi \in \Gamma_0(X)$ is a convex l.s.c. and proper function.

The coercive case

Several existence results for $V.I.(A, f, \Phi, K)$ are known when the operator A

(i) is **linear and coercive** , i.e. $\exists \alpha > 0$ such that :

$$\langle \mathbf{Au}, \mathbf{u} \rangle \geq \alpha \|\mathbf{u}\|^2, \ \forall \mathbf{u} \in \mathbf{X},$$

(ii) is **non-linear and coercive** i.e. :

$$\lim_{\|\mathbf{u}\| \to +\infty} \frac{\langle \mathbf{Au}, \mathbf{u} \rangle}{\|\mathbf{u}\|} = +\infty.$$

See the contributions of Brézis, Browder, J.L. Lions, Mosco, Stampacchia, Fichera etc ...

The non-coercive case

For the Non-coercive case, we refer to the works of :

- Fichera (1964)
- J.L. Lions & G. Stampacchia (1967)
- C. Baiocchi, F. Gastaldi, F. Tomarelli (1985)
- C. Baiocchi, G. Buttazzo, F. Gastaldi, F. Tomarelli (1988)
- F. Tomarelli (1993)
- D. Goeleven (1994)
- S. Adly, D. Goeleven, M. Théra (1996) etc ...
- A. Auslender (1996).

Motivation

Let us consider the following classical Neumann problem :

$$\mathcal{N}(f) \begin{cases} \text{find } u \in H^1(\Omega) \text{ such that} \\ -\Delta u = f, \text{ in } \Omega \\ \dfrac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega \end{cases}$$

It is well-known that $\mathcal{N}(f)$ has a solution if and only if

$$\int_\Omega f(x)dx = 0.$$

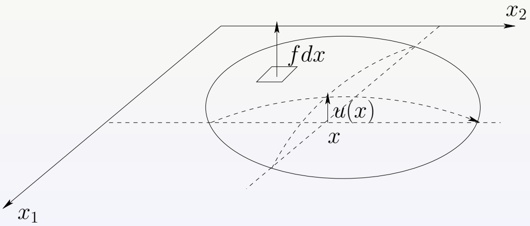
If we replace f by $f_\varepsilon = f + \varepsilon$ (with $\varepsilon > 0$), then $\mathcal{N}(f_\varepsilon)$ has no solutions.

The problem $\mathcal{N}(f)$ is unstable.

Obstacle problem without friction.

Let Ω be an open bounded subset of $\mathbb{R}^2$ (representing a thin elastic membrane).
For $f \in L^2(\Omega)$ and Ψ a given obstacle, we consider the following problem :

$$(\mathcal{P}) \begin{cases} -\Delta u \geq f, \text{ in } \Omega \\ (-\Delta u - f)(u - \Psi) = 0, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases}$$

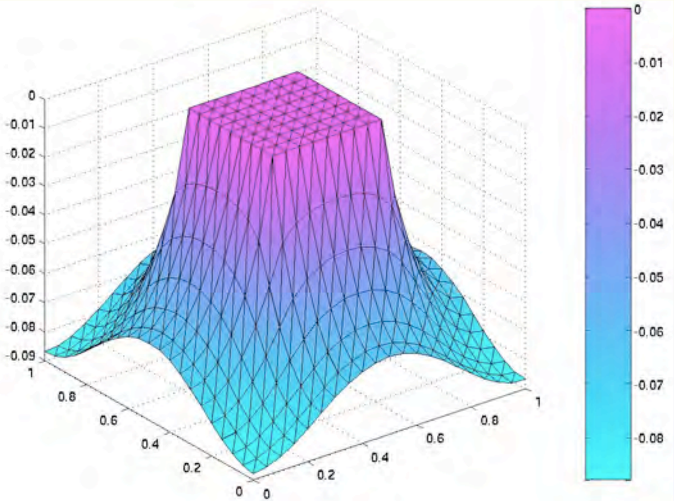


The variational formulation of the problem $(\mathcal{P})$ is :

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Find } u \in K \text{ such that} \\ \langle Au - f, v - u \rangle \geq 0, \quad \forall v \in K \end{array} \right.$$

- $X = H_0^1(\Omega)$
- $K = \{ v \in X \mid v \geq \Psi \}$
- $\langle Au, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$
- $\langle f, v \rangle = \int_\Omega f \, v dx$

If the condition $u = 0$ on $\partial\Omega$ is replaced by $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$ then the operator $A : X := H^1(\Omega) \to X'$ is no more coercive.



If the problem $(\mathcal{P})$ has a solution, then $\int_\Omega f dx \leq 0$.

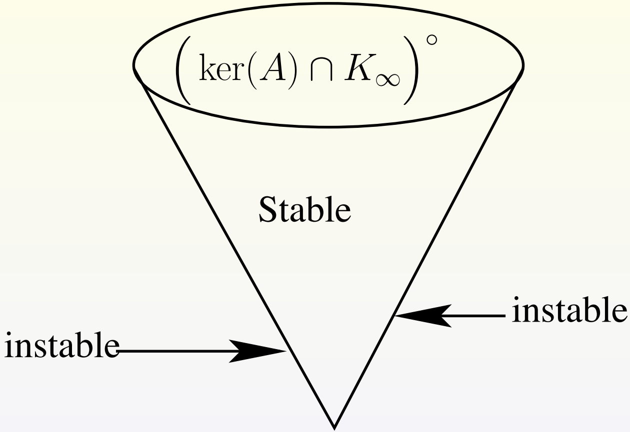
If $\int_\Omega f dx < 0$, then the problem has a solution.

Equivalently :

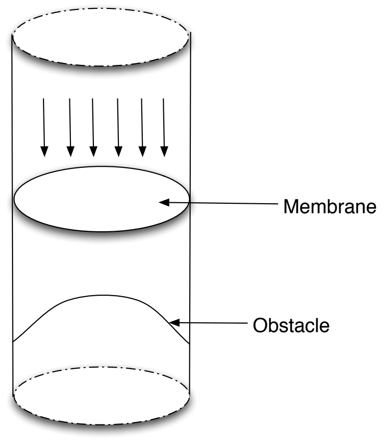
Necessary Condition : $f \in \left[\ker A \cap K_\infty\right]^\circ$.

Suffisante Condition : $f \in \mathrm{Int}\left([\ker A \cap K_\infty]^\circ\right)$.

The problem is unstable on the boundary.


$\left(\ker(A) \cap K_\infty\right)^\circ$
Stable
instable
instable

The puffed-up membrane



The puffed-up membrane

$$\begin{cases} \text{Find } u \in K = \{v \in W^{1,2}(\Omega) \ : \ v \geq \Psi \ \text{ on } \Omega\} \\ \int_\Omega \nabla u \cdot \nabla (v-u) dx + \int_{\partial\Omega} g(|v| - |u|) d\sigma \geq \int_\Omega f(v-u)dx, \\ \forall v \in K. \end{cases}$$

Necessary condition : $\int_{\partial\Omega} g d\sigma \geq \int_\Omega f dx.$

Suffisante condition : $\int_{\partial\Omega} g d\sigma > \int_\Omega f dx.$

Outline

We consider the following finite dimensional variational inequality

$$\mathrm{VI}(A, f, \varphi, K) \begin{cases} \text{Find } u \in K \text{ such that} \\ \langle Mu - q, v - u \rangle + \Phi(v) - \Phi(u) \geq 0, \ \forall v \in K \end{cases}$$

- $M \in \mathbb{R}^{n \times n}$ is a symmetric and positive semidefinite matrix ;
- $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, lower semicontinuous and bounded from below ;
- $K \subset \mathbb{R}^n$ is a closed and convex set such that $0 \in \mathrm{Dom}\ \Phi \cap K$.

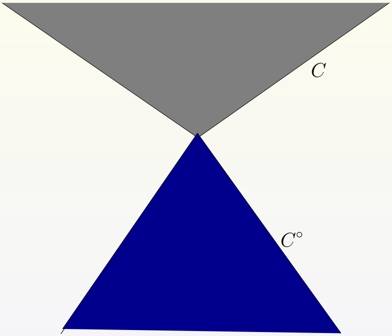
- The recession function :

$$\Phi_\infty(x) = \lim_{t \to +\infty} \frac{\Phi(x_0 + tx) - \Phi(x_0)}{t}, \ x_0 \in \mathrm{dom}(\Phi).$$

$$\Big(\mathrm{epi}\,(\Phi)\Big)_\infty = \mathrm{epi}\,(\Phi_\infty).$$

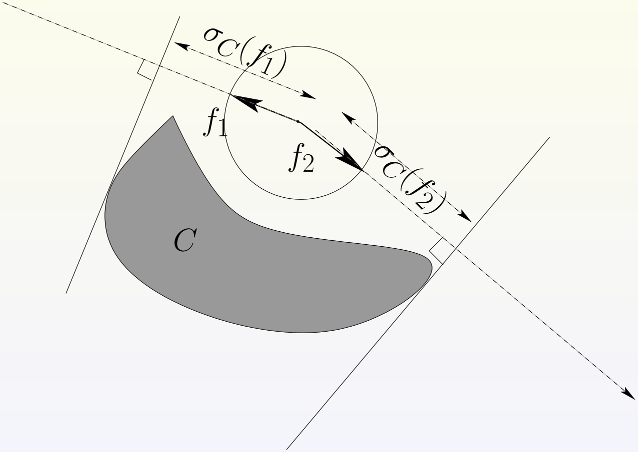
- The polar cone :

$$C^\circ = \{f \in X^* \ : \ \langle f, x\rangle \leq 0, \ \forall x \in C\}.$$

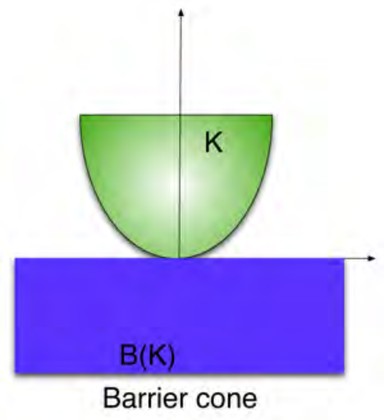


- The support function :

$$\sigma_C(f) = \sup_{x \in C} \langle f, x \rangle.$$

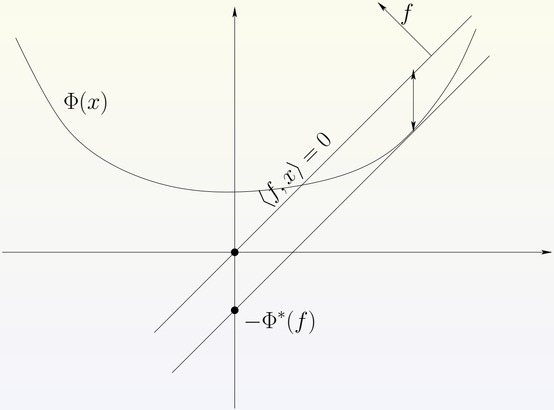


- The Barrier cone : $\mathcal{B}(C) = \text{dom } \sigma_C$.



- The Fenchel-Legendre conjugate :

$$\Phi^*(f) = \sup_{x \in X} \langle f, x \rangle - \Phi(x).$$



Let us finally recall the following proposition

Proposition

Let $\Psi \in \Gamma_0(\mathbb{R}^n)$ and $p \in \mathbb{R}^n$ be given. We have :

(i) $p \in \overline{\mathrm{Dom}\ \Psi^*} \Longleftrightarrow \Psi_\infty(w) \geq \langle p, w \rangle,\ \forall w \in \mathbb{R}^n;$

(ii) $p \in \mathrm{Int}\,(\mathrm{Dom}\ \Psi^*) \Longleftrightarrow \Psi_\infty(w) > \langle p, w \rangle,\ \forall w \in \mathbb{R}^n \backslash \{0\}.$

The solutions set of $\mathrm{VI}(A, f, \varphi, K)$ will be denoted by $\mathrm{Sol}(M, q, \Phi, K)$.
The following resolvent set will also play an important role

$$\mathcal{R}(A, \Phi, K) = \{q \in \mathbb{R}^n \ : \ \mathrm{Sol}(M, q, \Phi, K) \neq \emptyset\}.$$

Let us introduce the following function
$\Psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\Psi(u) = \frac{1}{2}\|Qu\|^2 + \Phi(u) + I_K(u), \tag{1}$$

where $Q = I - P_{\ker(M)}$ and $P_{\ker(M)}$ denotes the orthogonal projector from $\mathbb{R}^n$ to $\ker(M)$.

Let

$$\Psi(u) = \frac{1}{2}\|u - P_{\ker(M)}(u)\|^2 + \Phi(u) + I_K(u),$$

We have the following lemma :

Lemma

Suppose that the assumptions $(\mathcal{H})$ hold. We have

- $\mathrm{Dom}\,\Psi^* = R(M) + \mathrm{Dom}\left(\Phi + I_K\right)^*.$
- $\Psi_\infty(w) = I_{\ker(M)}(w) + \Phi_\infty(w) + I_{K_\infty}(w),$

Proposition

A necessary condition for the existence of a solution of $\mathrm{VI}(A, f, \varphi, K)$ *is that*

$$\Phi_\infty(w) \geq \langle q, w \rangle, \ \forall w \in \ker(M) \cap K_\infty. \tag{2}$$

Proposition

Suppose that assumptions $(\mathcal{H})$ hold. We have

$$\begin{aligned} \operatorname{Int}\mathcal{R}(M,\Phi,K) &= \Big\{ q\in\mathbb{R}^n \ : \ \Phi_\infty(w) > \langle q, w\rangle, \\ &\qquad \forall w\in \ker(M)\cap K_\infty\setminus\{0\}\Big\} \end{aligned}$$

Definition

The variational inequality $\mathrm{VI}(A, f, \varphi, K)$ is stable if there exists $\varepsilon > 0$ such that for any $M_\varepsilon \in \mathcal{S}_n^+(\mathbb{R})$, any vector $q_\varepsilon \in q + \varepsilon\mathbb{B}_n$, any $\Phi_\varepsilon \in \Gamma_0(\mathbb{R}^n)$ bounded from below, and any nonempty closed convex set K_ε satisfying the following conditions

$$0 \in \mathrm{Dom}\ \Phi_\varepsilon \cap K_\varepsilon$$

$$\ker(M) \cap \ker(\Phi_\infty) \cap K_\infty = \ker(M_\varepsilon) \cap \ker((\Phi_\varepsilon)_\infty) \cap (K_\varepsilon)_\infty$$

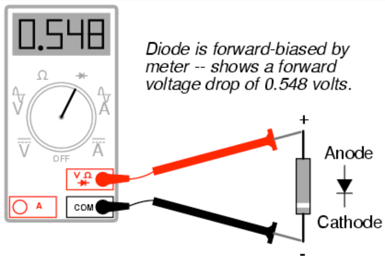
the perturbed problem $VI(M_\varepsilon, q_\varepsilon, \Phi_\varepsilon, K_\varepsilon)$ has at least one solution.

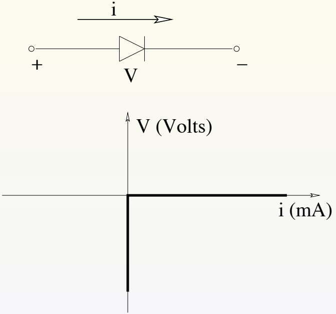
Theorem

Assume that assumptions $(\mathcal{H})$ are satisfied. Then the variational inequality $\mathrm{VI}(A, f, \varphi, K)$ is stable in the sense of Definition 2 if and only if

$$\Phi_\infty(w) > \langle q, w\rangle, \ \forall w \in \mathrm{ker}(M) \cap K_\infty, \ w \neq 0.$$

Ideal diode Model





Ideal Diode

- $V < 0, i = 0 \implies$ diode is blocking ;
- $i > 0, V = 0 \implies$ diode is conducting ;

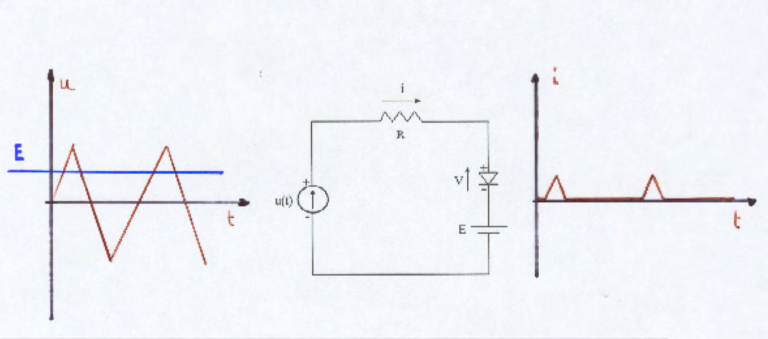
Complementarity formulation

$$V \leq 0, \quad i \geq 0, \quad Vi = 0 \quad \Longleftrightarrow \quad \min\{-V, i\} = 0$$

$$V \in \partial \Psi_{\mathbb{R}_+}(i) \quad \Longleftrightarrow \quad i \in \partial \Psi^*_{\mathbb{R}_+}(V) = \partial \Psi_{\mathbb{R}_-}(V)$$

$$\Psi_{\mathbb{R}_+}(x) := \begin{cases} 0 \ \text{ if } \ x \geq 0 \\ +\infty \ \text{ if } \ x < 0 \end{cases} \qquad \partial \Psi_{\mathbb{R}_+}(x) := \begin{cases} \mathbb{R}_- \ \text{ if } \ x = 0 \\ 0 \ \text{ if } \ x > 0 \\ \emptyset \ \text{ if } \ x < 0 \end{cases}$$

Complementarity formulation



$$u = \overbrace{U_R}^{Ri} + \overbrace{V}^{\in\ \partial\Psi_{\mathbb{R}_+}(i)} + E \iff E + Ri - u \in -\partial\Psi_{\mathbb{R}_+}(i)$$

Complementarity formulation

$$\frac{E}{R} + i - \frac{u}{R} \in -\partial\Psi_{\mathbb{R}_+}(i) \iff -\frac{E}{R} + \frac{u}{R} \in i + \partial\Psi_{\mathbb{R}_+}(i)$$

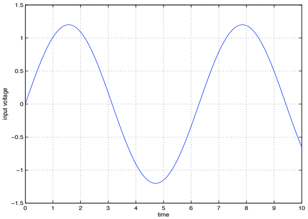
$$i = (id + \partial\Psi_{\mathbb{R}_+})^{-1}(\frac{u - E}{R}) = \frac{1}{R}\max\{0, u - E\}$$

$$u < E \implies \text{diode is blocking}$$

$$u \geq E \implies \text{diode is conducting}$$

Numerical simulation : Ideal diode

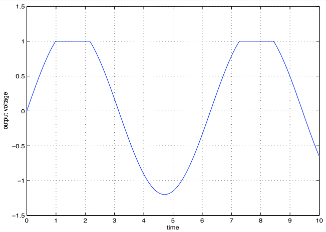
INPUT SIGNAL $t \mapsto u(t)$



Numerical simulation : Ideal diode

OUTPUT SIGNAL $t \mapsto V_o(t) := V(t) + E = u(t) - Ri(t)$

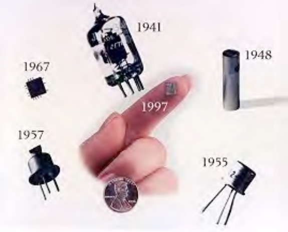
$$= u(t) - R\max\{0, \frac{u(t) - E}{R}\} = u(t) + \min\{0, E - u(t)\}$$
$$= \min\{u(t), E\}$$

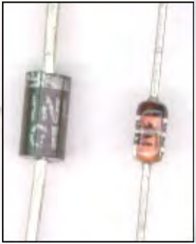


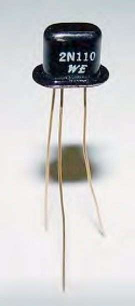
Outline

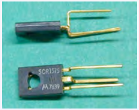
Set-Valued Ampere-Volt characteristic in electronics

- Diode
- Zener Diode
- Varactor
- Triode
- Tetrode
- Transistor
- Diac
- Triac
- Silicon Controlled Rectifier

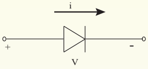


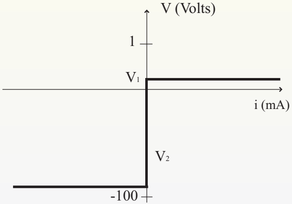






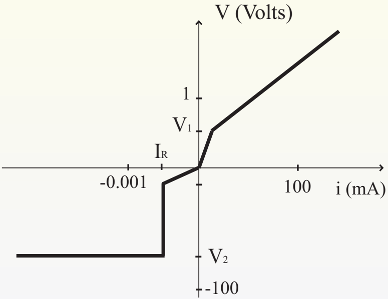
Set-Valued Ampere-Volt characteristic : the Diode model





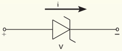
There is a voltage point, called the knee voltage V_1, at which the diode begins to conduct and a maximum reverse voltage, called the peak reverse voltage V_2,that will not force the diode to conduct.

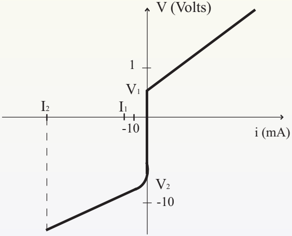
Set-Valued Ampere-Volt characteristic : the complete diode model



Illustrates a complete diode model which includes the effects of the natural resistance of the diode

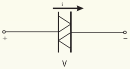
Set-Valued Ampere-Volt characteristic : the Zener diode model

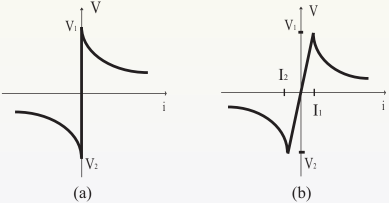




Zener diode is a good voltage regulator to maintain a constant voltage regardless of minor variations in load current or input voltage.

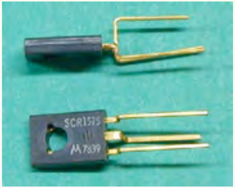
Set-Valued Ampere-Volt characteristic in electronics

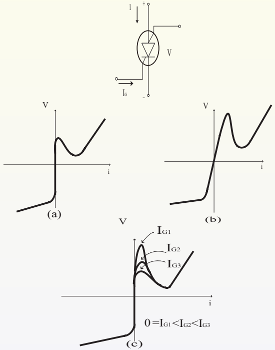




Illustrates a typical voltage current characteristics of a diac.

Set-Valued Ampere-Volt characteristic in electronics

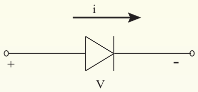


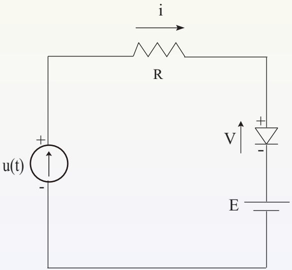


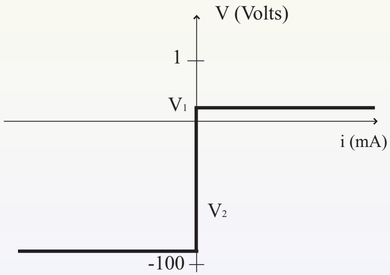
Illustrates the AV-characteristic of a three-terminal silicon controlled rectifier which is used for start/stop control circuit for a direct current motor, lamp...

Outline

Clipping circuit







The electrical superpotential of the practical diode is

$$\varphi_{PD}(x) = \begin{cases} V_1 x & \text{if } x \geq 0 \\ V_2 x & \text{if } x < 0 \end{cases}, \quad (x \in \mathbb{R}).$$

Then

$$\varphi^*_{PD}(z) = I_{[V_2, V_1]}(z), \quad (z \in \mathbb{R})$$

We see that

$$\partial\varphi_{PD}(x) = \begin{cases} V_2 & \text{if } x < 0 \\ [V_2, V_1] & \text{if } x = 0 \\ V_1 & \text{if } x > 0 \end{cases}, \quad (x \in \mathbb{R})$$

$$\partial\varphi^*_{PD}(z) = \begin{cases} \mathbb{R}_- \ \text{ if } \ z = V_2 \\ \\ 0 \ \text{ if } \ z \in]V_2, V_1[\\ \\ \mathbb{R}_+ \ \text{ if } \ z = V_1 \\ \\ \emptyset \ \text{ if } \ z \in \mathbb{R} \setminus [V_2, V_1] \end{cases}, \quad (z \in \mathbb{R}).$$

recovers the volt-ampere characteristic (V, i). The ampere-volt characteristic of the practical diode can thus be written as

$$V \in \partial\varphi_{PD}(i) \iff i \in \partial\varphi^*_{PD}(V) \iff \varphi_{PD}(i) + \varphi^*_{PD}(V) = iV.$$

Using Kirchhoff's law the problem is equivalent to $\mathbf{VI}(\mathbf{R}, \mathbf{E} - \mathbf{u}, \varphi_{\mathbf{PD}}, \mathbb{R})$, i.e.

$$i \in K := \mathbb{R} : (Ri + E - u)(v - i) + \varphi_{PD}(v) - \varphi_{PD}(i) \geq 0, \forall v \in \mathbb{R}.$$

Here $R > 0$ and for each $E, u \in \mathbb{R}$.

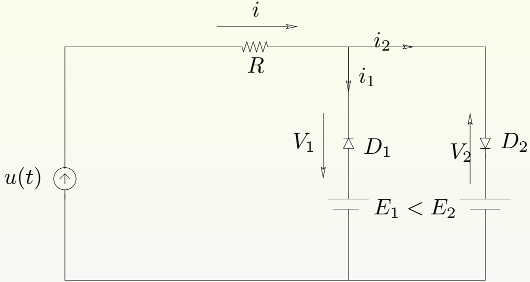
Moreover :

$$\begin{aligned} i(t) &= (id_{\mathbb{R}} + \partial\varphi_{PD})^{-1}\left(\frac{u(t)-E}{R}\right) \\ &= \operatorname{argmin}_{x\in\mathbb{R}}\left\{\frac{1}{2}\left|x-\left(\frac{u(t)-E}{R}\right)\right|^2 + \varphi_{PD}(x)\right\}. \end{aligned}$$

and

$$V_o(t) = u(t) - Ri(t). \tag{3}$$

A static model



KIRCHOFF'S LAWS

$$E_1 + R(i_1 + i_2) - u = +V_1 \in -\partial\Psi_{\mathbb{R}_-}(i_1)$$

$$E_2 + R(i_1 + i_2) - u = -V_2 \in -\partial\Psi_{\mathbb{R}_+}(i_2)$$

Non-coercive complementarity formalism

$$\overbrace{\begin{pmatrix} R & R \\ R & R \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} i_1 \\ i_2 \end{pmatrix}}^{I} + \overbrace{\begin{pmatrix} E_1 - u \\ E_2 - u \end{pmatrix}}^{U} \in -\partial \Psi_{\mathbb{R}_- \times \mathbb{R}_+}(I).$$

Here the matrix A is positive semidefinite and symmetric. A sufficient condition for the existence of at least one solution is :

$$\langle q, v \rangle > 0, \ \forall v \in \ker\{A\} \cap (\mathbb{R}_- \times \mathbb{R}_+) \setminus \{0\}.$$

Non-coercive complementarity formalism

$$\langle q, v \rangle > 0, \ \forall v \in \ker\{A\} \cap (\mathbb{R}_- \times \mathbb{R}_+) \setminus \{0\}.$$

We have

$$\ker\{A\} \cap (\mathbb{R}_- \times \mathbb{R}_+) = \{v = (v_1, v_2) \in \mathbb{R}^2 : v_1 \leq 0, v_2 = -v_1\}.$$

Then, for all $v \in \ker\{A\} \cap \mathbb{R}_- \times \mathbb{R}_+$, $v \neq 0$, we get

$$q^T v = (E_1 - u)v_1 + (E_2 - u)v_2 = v_2(E_2 - E_1) > 0.$$

Then I is determined as the unique solution of the complementarity system :

$$I \in K, \quad AI + U \in K^*, \quad I^T(AI + U) = 0$$

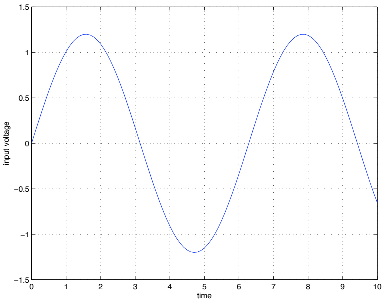
$$\Updownarrow$$

VARIATIONAL INCLUSION SYSTEM

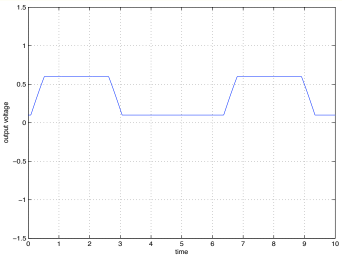
$$AI + U \in -\partial\Psi_{\mathbb{R}_+ \times \mathbb{R}_+}(I)$$

Moreover

$$i_1 + i_2 = \begin{cases} \dfrac{u - E_1}{R} \text{ if } u < E_1 \\ 0 \text{ if } E_1 \leq u \leq E_1 \\ \dfrac{u - E_2}{R} \text{ if } u > E_2 \end{cases}.$$



input voltage
time



output voltage
time

Non-regular electronic circuits

- $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{n\times p}$,
- $j : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, $x \mapsto j(t,x)$ locally Lipschitz for all $t \geq 0$,
- $u \in L^1_{loc}(0,+\infty;\mathbb{R}^p)$, $x_0 \in \mathbb{R}^n$.

Problem $P(x_0)$ **:** Find $x : [0,+\infty[\to \mathbb{R}^n$; and $y_L : [0,+\infty[\to \mathbb{R}^m$ such that :

$$x \in C^0([0,+\infty[;\mathbb{R}^n), \quad By_L \in L^1_{loc}(0,+\infty;\mathbb{R}^n),$$

$$\frac{dx}{dt} \in L^1_{loc}(0,+\infty;\mathbb{R}^n), \quad x(0) = x_0,$$

$$\frac{dx}{dt}(t) = Ax(t) - By_L(t) + Du(t), \text{ a.e. } t \geq 0,$$

$$y(t) = Cx(t), \ \forall\, t \geq 0, \text{ and } y_L(t) \in \partial_2 j(t, y(t)), \text{ a.e. } t \geq 0.$$

Non-regular circuit with VAP-admissible device

(H) There exists a symmetric and invertible matrix $R \in \mathbb{R}^{n\times n}$ such that

$$R^{-2}C^T = B.$$

Problem $Q(x_0)$ **:** Find $z : [0, +\infty[\to \mathbb{R}^n;\ t \mapsto z(t)$ such that :

$$z \in C^0([0, +\infty[; \mathbb{R}^n), \ \ \frac{dz}{dt} \in L^1_{loc}(0, +\infty; \mathbb{R}^n), \ \ z(0) = Rx_0,$$

$$\frac{dz}{dt}(t) \in RAR^{-1}z(t) + RDu(t) - R^{-1}C^T\partial_2 j(t, CR^{-1}z(t)).$$

Non-regular circuit with VAP-admissible device

Suppose that assumption (H) is satisfied.

If (x, y_L) is solution of Problem $P(x_0)$ then $z = Rx$ is solution of Problem $Q(x_0)$.

Reciprocally, if z is solution of Problem $Q(x_0)$ then there exists a function y_L such that $(R^{-1}z, y_L)$ is solution of Problem $P(x_0)$.

Non-regular circuit with VAP-admissible device

Set

$$L = CR^{-1}.$$

and

$$J(t, X) = j(t, LX), \ (X \in \mathbb{R}^n, \ t \geq 0)$$

then

$$\partial_2 J(t, X) \subset R^{-1} C^T \partial_2 j(t, CR^{-1} X), \ (X \in \mathbb{R}^n, \ t \geq 0)$$

$$\Downarrow$$

DIFFERENTIAL INCLUSION

$$\frac{dz}{dt}(t) \in RAR^{-1} z(t) + RDu(t) - \partial_2 J(t, z(t)), \text{ a.e. } t \geq 0$$

Non-regular circuit with VAP-admissible device

DIFFERENTIAL INCLUSION

$$\frac{dz}{dt}(t) \in RAR^{-1}z(t) + RDu(t) - \partial_2 J(t, z(t)), \text{ a.e. } t \geq 0$$

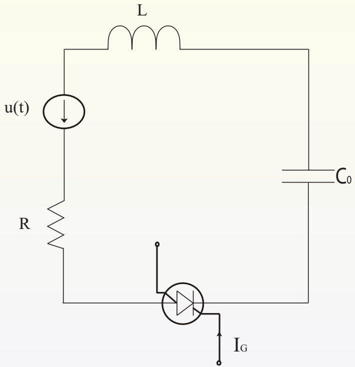
$$\Downarrow$$

VARIATIONAL INEQUALITY

$$\langle \frac{dz}{dt}(t) - RAR^{-1}z(t) - RDu(t), v - z(t) \rangle +$$

$$+ J(t, v) - J(t, z(t)) \geq 0, \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq 0.$$

Example



Example

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -\frac{1}{LC_0} & -\frac{R}{L} \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \overbrace{\begin{pmatrix} 0 \\ -\frac{1}{L} \end{pmatrix}}^{B} y_L + \overbrace{\begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix}}^{D} u,$$

$$y = \overbrace{\begin{pmatrix} 0 & -1 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$y_L \in \partial_{C,2} j_{SCR}(., y).$$

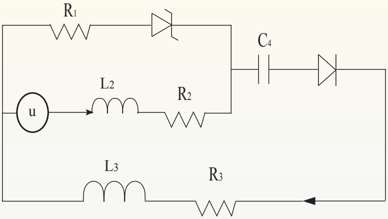
(H) There exists a symmetric and invertible matrix $R \in \mathbb{R}^{n \times n}$ such that

$$R^{-2} C^T = B.$$

$$R = \begin{pmatrix} \frac{1}{\sqrt{C_0}} & 0 \\ 0 & \sqrt{L} \end{pmatrix}$$

Hence assumption (H) is satisfied.

Example 2



$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{L_3 C_4} & -\frac{(R_1+R_3)}{L_3} & \frac{R_1}{L_3} \\ 0 & \frac{R_1}{L_2} & -\frac{(R_1+R_2)}{L_2} \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$- \overbrace{\begin{pmatrix} 0 & 0 \\ \frac{1}{L_3} & \frac{1}{L_3} \\ -\frac{1}{L_2} & 0 \end{pmatrix}}^{B} \begin{pmatrix} y_{L,1} \\ y_{L,2} \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 0 \\ \frac{1}{L_2} \end{pmatrix}}^{D} u,$$

$$\begin{cases} y_{L,1} \in \partial j_D(-x_3 + x_2) \\ y_{L,2} \in \partial j_Z(x_2) \end{cases} \tag{4}$$

Setting

$$y = \overbrace{\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$j_{ZD}(X) = j_Z(X_1) + j_D(X_2),\ (X \in \mathbb{R}^2)$$

$$\begin{cases} y_{L,1} \in \partial j_D(-x_3 + x_2) \\ y_{L,2} \in \partial j_Z(x_2) \end{cases} \iff y_L \in \partial j_{ZD}(Cx).$$

A simple computation shows that the matrix

$$R = \begin{pmatrix} \frac{1}{\sqrt{C_4}} & 0 & 0 \\ 0 & \sqrt{L_3} & 0 \\ 0 & 0 & \sqrt{L_2} \end{pmatrix}$$

is convenient.
Hence assumption (H) is satisfied.