The Heat Flow on Metric Random Walk Spaces

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Outline

- Metric random walk spaces
- $m$-Heat flow
- $r$-connectedness
- Functional inequalities and curvature
- Other diffusion operators
Let $(X, d)$ be a Polish metric space equipped with its Borel $\sigma$-algebra.

**Definition**

A metric random walk space $[X, d, m]$ is a Polish metric space $(X, d)$ equipped with a random walk $m = (m_x)_{x \in X}$, a family of probability measures $m_x$ on $X$ given for each $x \in X$ and satisfying

(i) the measures $m_x$ depend measurably on the point $x \in X$,

(ii) each measure $m_x$ has finite first moment, i.e., for some (hence any) $z \in X$, $\int_X d(z, y)dm_x(y) < +\infty$.

A Radon measure $\nu$ on $X$ is **invariant** for the random walk $m = (m_x)$ if

$$d\nu(x) = \int_{y \in X} d\nu(y)dm_y(x).$$

$\nu$ is said to be **reversible** if

$$dm_x(y)d\nu(x) = dm_y(x)d\nu(y).$$
Example

Let \((\mathbb{R}^N, d, \mathcal{L}^N)\), with \(d\) the Euclidean distance and \(\mathcal{L}^N\) the Lebesgue measure. Let \(J : \mathbb{R}^N \to [0, +\infty]\) be a measurable, nonnegative and radially symmetric function verifying \(\int_{\mathbb{R}^N} J(z)dz = 1\). In \((\mathbb{R}^N, d, \mathcal{L}^N)\) we can give the following random walk, starting at \(x\),

\[
m^J_x(A) = \int_A J(x - y)d\mathcal{L}^N(y) \quad \forall A \subset \mathbb{R}^N \text{ borelian.}
\]

The Lebesgue measure \(\mathcal{L}^N\) is a reversible measure for this random walk.
Example

Let $K : X \times X \to \mathbb{R}$ be a Markov kernel on a countable space $X$, i.e.,

$$K(x, y) \geq 0, \quad \forall x, y \in X, \quad \sum_{y \in X} K(x, y) = 1 \quad \forall x \in X.$$

Then, for

$$m^K_x(A) = \sum_{y \in A} K(x, y),$$

$[X, d, m^K]$ is a metric random walk for any metric $d$ on $X$, and the steady state $\pi$ on $X$ is invariant for $m^K$. 
A weighted discrete graph $G = (V(G), E(G))$ is a graph of vertices $V(G)$ and edges $E(G)$ such that a positive weight $w_{xy} = w_{yx}$ is assigned to each edge $(x, y) \in E(G)$ (we will write $x \sim y$ if $(x, y) \in E(G)$). We consider that $w_{xy} = 0$ if $(x, y) \notin E(G)$. 

For each $x \in V(G)$, define the following probability measure $m_{Gx} = \frac{1}{d_x} \sum_{y \sim x} w_{xy} \delta_y$ with $d_x = \sum_{y \sim x} w_{xy}$. Then $\left[ V(G), d_G, (m_{Gx}) \right]$, for the graph distance $d_G$, is a metric random walk space. And the measure $\nu_G$ defined as $\nu_G(A) = \sum_{x \in A} d_x$, $A \subset V(G)$ is a reversible measure for this random walk.
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**Example**

Let $G = (V(G), E(G))$ be a locally finite weighted connected graph. For each $x \in V(G)$, define the following probability measure

$$m^G_x = \frac{1}{d_x} \sum_{y \sim x} w_{xy} \delta_y \quad \text{with} \quad d_x = \sum_{y \sim x} w_{xy}.$$
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Then $[V(G), d_G, (m^G_x)]$, for the graph distance $d_G$, is a metric random walk space. And the measure $\nu_G$ defined as

$$\nu_G(A) = \sum_{x \in A} d_x, \quad A \subset V(G)$$

is a reversible measure for this random walk.
Metric random walk spaces

Example

From a metric measure space \((X, d, \mu)\) we can obtain a metric random walk space with the \(\epsilon\)-step random walk associated to \(\mu\).
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Assume that balls in \(X\) have finite measure and that \(\text{Supp}(\mu) = X\). Given \(\epsilon > 0\), the \(\epsilon\)-step random walk on \(X\), starting at point \(x\), consists in randomly jumping in the ball of radius \(\epsilon\) around \(x\), with probability proportional to \(\mu\):

\[
m^\mu_{\epsilon} = \frac{\mu \llcorner B(x, \epsilon)}{\mu(B(x, \epsilon))}.
\]

We have that \(\mu\) is a reversible measure for \(m^\mu_{\epsilon}\).
Example (6)

Given a metric random walk space \([X, d, m]\) with reversible measure \(\nu\) for \(m\), and given a \(\nu\)-measurable set \(\Omega \subset X\) with \(\nu(\Omega) > 0\), if we define, for \(x \in \Omega\),

\[
m^\Omega_x(A) = \int_A dm_x(y) + \left(\int_{X \setminus \Omega} dm_x(y)\right) \delta_x(A) \quad \forall A \subset \Omega \text{ borelian,}
\]

we have that \([\Omega, d, m^\Omega]\) is a metric random walk space, and \(\nu_{\Omega}\) is reversible for \(m^\Omega\).
Given a metric random walk space \([X, d, m]\), the transition probability from \(x\) to \(y\) in \(n\) steps is

\[
dm_x^n(y) = \int_{z \in X} dm_z(y) dm_x^{(n-1)}(z)
\]

where \(m_x^1 = m_x\).
Let \([X, d, m]\) be a metric random walk space with invariant measure \(\nu\) for \(m\). For a function \(u : X \to \mathbb{R}\) we define its nonlocal gradient 
\[
\nabla u : X \times X \to \mathbb{R}
\]
as
\[
\nabla u(x, y) = u(y) - u(x) \quad \forall x, y \in X,
\]
The heat flow

Let \([X, d, m]\) be a metric random walk space with invariant measure \(\nu\) for \(m\). For a function \(u : X \to \mathbb{R}\) we define its nonlocal gradient \(\nabla u : X \times X \to \mathbb{R}\) as

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\nabla u(x, y) = u(y) - u(x) \quad \forall x, y \in X,
\]

For a function \(z : X \times X \to \mathbb{R}\), its \(m\)-divergence \(\text{div}_m z : \mathbb{R}^N \to \mathbb{R}\) is defined as

\[
(\text{div}_m z)(x) = \frac{1}{2} \int_X (z(x, y) - z(y, x)) dm_x(y).
\]
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The \(m\)-Laplace operator is defined as

\[
\Delta_m f(x) = \int_X f(y)dm_x(y) - f(x) = \int_X (f(y) - f(x))dm_x(y).
\]

\[
\Delta_m f(x) = \text{div}_m(\nabla f)(x)
\]
The heat flow

For the energy functional in $L^2(X, \nu)$,

$$
\mathcal{H}_m(f) = \begin{cases} 
\frac{1}{2} \int_{X \times X} (f(x) - f(y))^2 \, dm_x(y) \, d\nu(x) & \text{if } f \in L^2(X, \nu) \cap L^1(X, \nu), \\
+\infty, & \text{else},
\end{cases}
$$

$\partial \mathcal{H}_m = -\Delta_m$ is a maximal monotone operator in $L^2(X, \nu)$.
Moreover, $-\Delta_m$ is completely accretive operator.
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**Theorem**

Let $[X, d, m]$ be a metric random walk space with reversible measure $\nu$ for $m$. Then, $-\Delta_m$ is a non-negative self-adjoint operator in $L^2(X, \nu)$ that generates a strongly continuous Markovian semigroup, $(e^{t\Delta_m})_{t \geq 0}$, called the $m$-heat flow on the metric random walk space $[X, d, m]$. 
For every $u_0 \in L^2(X, \nu)$, $u(t) = e^{t\Delta_m} u_0$ is the unique strong solution of the $m$-heat equation

$$\begin{cases}
\frac{du}{dt} = \Delta_m u(t) & \text{in } (0, T) \times X, \\
u(0) = u_0.
\end{cases}$$

and

$$\|e^{t\Delta_m} u_0\|_{L^p(X, \nu)} \leq \|u_0\|_{L^p(X, \nu)} \quad \forall u_0 \in L^p(X, \nu) \cap L^2(X, \nu), \quad 1 \leq p \leq +\infty.$$
The heat flow

Example

For $\mathbb{R}^N$ with $m^J_x$, $u(t) = e^{t\Delta m^J} u_0$ is the solution of

\[
\begin{cases}
  \frac{du}{dt}(t, x) = \int_{\mathbb{R}^N} J(x - y)(u(t, y) - u(t, x))dy & \text{in } \mathbb{R}^N \times (0, +\infty), \\
  u(0) = u_0.
\end{cases}
\]

And, for $\Omega$ a domain in $\mathbb{R}^N$, and $m^J_x, \Omega$ (Example(6) above),

$u(t) = e^{t\Delta m^J, \Omega} u_0$ is the solution of the $J$-Heat equation in $\Omega$

with homogeneous Neumann boundary condition:

\[
\begin{cases}
  \frac{du}{dt}(t, x) = \int_{\Omega} J(x - y)(u(t, y) - u(t, x))dy & \text{in } \Omega \times (0, +\infty), \\
  u(0) = u_0;
\end{cases}
\]
The heat flow

For a given a metric random walk space $[X, d, m]$ with reversible measure $\nu$, and a given $\nu$-measurable set $\Omega \subset X$ with $\nu(\Omega) > 0$, and for $[\Omega, d, m^\Omega]$ the metric random walk space with

$$m^\Omega_x(A) = \int_A dm_x(y) + \left( \int_{X \setminus \Omega} dm_x(y) \right) \delta_x(A) \quad \forall A \subset \Omega \text{ borelian},$$

for each $x \in \Omega$,

$$u(t) = e^{t\Delta_{m^\Omega}} u_0 \text{ is the solution of}$$

$$\begin{cases}
\frac{du}{dt}(t, x) = \int_{\Omega} (u(t, y) - u(t, x)) dm_x(y) & \text{in } \Omega \times (0, +\infty),
\end{cases}$$

$$u(0) = u_0,$$

which is the $m$-heat equation in $\Omega$ with homogeneous Neumann boundary condition.
The heat flow

**Theorem**

Let \([X, d, m]\) be a metric random walk with reversible measure \(\nu\) for \(m\). Let \(u_0 \in L^2(X, \nu) \cap L^1(X, \nu)\). Then,

\[
e^{t\Delta_m} u_0(x) = e^{-t} \sum_{n=0}^{\infty} \int_X u_0(y) dm_x^n(y) t^n n!,
\]

where \(\int_X u_0(y) dm_x^0(y) = u_0(x)\).
Infinite speed of propagation and \( r \)-connectedness

Let \([X, d, m]\) be a metric random walk with invariant measure \(\nu\). For a \(\nu\) measurable set \(D\), we set

\[
N^m_D = \{x \in X : m_x^*(D) = 0 \ \forall n \in \mathbb{N}\}.
\]

**Definition**

A metric random walk space \([X, d, m]\) with invariant measure \(\nu\) is called random-walk-connected (\(r\)-connected) if for any \(D \subset X\) with \(0 < \nu(D) < +\infty\) we have that \(\nu(N^m_D) = 0\).
Let \([X, d, m]\) be a metric random walk with invariant measure \(\nu\). For a \(\nu\) measurable set \(D\), we set

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A metric random walk space \([X, d, m]\) with invariant measure \(\nu\) is called random-walk-connected (\(r\)-connected) if for any \(D \subset X\) with \(0 < \nu(D) < +\infty\) we have that \(\nu(N^m_D) = 0\).

**Theorem**

Let \([X, d, m]\) be a metric random walk with reversible measure \(\nu\). The space is \(r\)-connected if and only if, for any non-null \(0 \leq u_0 \in L^2(X, \nu)\), we have \(e^{t\Delta_m}u_0 > 0\) \(\nu\)-a.e. for all \(t > 0\), that is, the \(m\)-heat flow has infinite speed of propagation.
Infinite speed of propagation and $r$-connectedness

**Example**

Let $[V(G), d_G, (m^G_x)]$ be the random walk metric space associated with a locally finite weighted connected graph $G = (V(G), E(G))$. Then $[V(G), d_G, m^G]$ with $\nu_G$ is strong $r$-connected, that is, $\mathcal{N}_D^m = \emptyset$, which is equivalent to

$$e^{t\Delta_m} u_0(x) > 0 \quad \text{for all } x \in X, \text{ and for all } t > 0.$$
In Riemannian geometry, positive Ricci curvature is characterized by the fact that “small balls are closer, in the 1-Wasserstein distance, than their centers are”. In the framework of metric random walk spaces, inspired by this, Y. Ollivier [J. Funct. Anal. (2009)] introduces the concept of coarse Ricci curvature changing the balls by the measures $m_x$. 
Ollivier-Ricci curvature

In Riemannian geometry, positive Ricci curvature is characterized by the fact that “small balls are closer, in the 1-Wasserstein distance, than their centers are”. In the framework of metric random walk spaces, inspired by this, Y. Ollivier [J. Funct. Anal. (2009)] introduces the concept of coarse Ricci curvature changing the balls by the measures $m_x$.

Recall that for $(X, d)$ a Polish metric space and $\mathcal{M}^+(X)$ the positive Radon measures on $X$, and for $\mu, \nu \in \mathcal{M}^+(X)$ satisfying $\mu(X) = \nu(X)$, the 1-Wasserstein distance is the optimal cost of the Monge-Kantorovich problem, that is,

$$\min \left\{ \int_{X \times X} d(x, y) \, d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\},$$

where $\Pi(\mu, \nu)$ is the set of transport plans between $\mu$ and $\nu$. 

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Ollivier-Ricci curvature

that is, 1-Wasserstein distance between $\mu, \nu$ is

$$W_1^d(\mu, \nu) = \min \left\{ \int_{X \times X} d(x, y) \, d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\}.$$
Ollivier-Ricci curvature

that is, 1-Wasserstein distance between $\mu, \nu$ is

$$W^d_1(\mu, \nu) = \min \left\{ \int_{X \times X} d(x, y) \, d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\}.$$ 

**Definition (Ollivier)**

On a given metric random walk space $[X, d, m]$, for any two distinct points $x, y \in X$, the Ollivier-Ricci curvature of $[X, d, m]$ along $(x, y)$ is defined as

$$\kappa_m(x, y) = 1 - \frac{W^d_1(m_x, m_y)}{d(x, y)}.$$ 

The Ollivier-Ricci curvature of $[X, d, m]$ is defined by

$$\kappa_m = \inf_{x, y \in X \atop x \neq y} \kappa_m(x, y).$$

We will write $\kappa(x, y)$ instead of $\kappa_m(x, y)$, and $\kappa = \kappa_m$, if the context allows no confusion.
Example

Take $(\mathbb{R}, d)$ with $d$ the euclidean distance and let $C$ the Cantor set. Let $\mu$ be the Cantor distribution. We denote $\eta = \mathcal{L}^1 | [0, 1]$ and define the random walk

$$m_x = \begin{cases} 
\eta & \text{if } x \in [0, 1] \setminus C, \\
\mu & \text{if } x \in C.
\end{cases}$$

Then $\nu = \eta + \mu$ is reversible for $m$.

The space $([0, 1], d, m)$ is not $r$-connected.

Its Ollivier-Ricci curvature is $\kappa = -\infty$. 
Infinite speed of propagation and $r$-connectedness

Example

Let $\Omega = \left( -\infty, 0 \right] \cup \left[ \frac{1}{2}, +\infty \right[ \times \mathbb{R}^{N-1}$ and consider the metric random walk space $[\Omega, d, m^{J,\Omega}]$, with $d$ the Euclidean distance and $J(x) = \frac{1}{|B_1(0)|} \chi_{B_1(0)}$. It is easy to see that this space with reversible measure $\nu = \mathcal{L}\llcorner\Omega$ is $r$-connected but $(\Omega, d)$ is not connected. Its Ollivier-Ricci curvature is $\kappa < 0$.

For $\Omega = \left( -\infty, 0 \right] \cup \left[ 2, +\infty \right[ \times \mathbb{R}^{N-1}$, neither $[\Omega, d, m^{J,\Omega}]$ with $\nu = \mathcal{L}\llcorner\Omega$ is $r$-connected, nor $(\Omega, d)$ is connected. Its Ollivier-Ricci curvature is $\kappa < 0$. 

Theorem

Let $[\mathcal{X}, d, m]$ be a metric random walk space with invariant measure $\nu$ such that $\nu(\mathcal{X}) < +\infty$. Assume that the Ollivier-Ricci curvature $\kappa > 0$. Then, $[\mathcal{X}, d, m]$ with $\nu$ is $r$-connected.
Infinite speed of propagation and $r$-connectedness

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Definition

Let $[X, d, m]$ be a metric random walk space with invariant probability measure $\nu$. A Borel set $B \subseteq X$ is said to be invariant with respect to the random walk $m$ if $m_x(B) = 1$ whenever $x$ is in $B$.

The invariant probability measure $\nu$ is said to be ergodic if $\nu(B) = 0$ or $\nu(B) = 1$ for every invariant set $B$ with respect to $m$. 
Infinite speed of propagation and \( r \)-connectedness

**Definition**

Let \([X, d, m]\) be a metric random walk space with invariant probability measure \(\nu\). A Borel set \(B \subset X\) is said to be **invariant** with respect to the random walk \(m\) if \(m_x(B) = 1\) whenever \(x\) is in \(B\).

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**Theorem**

Let \([X, d, m]\) be a metric random walk with invariant probability measure \(\nu\). Then,

\[
\nu \text{ is ergodic } \iff [X, d, m] \text{ with } \nu \text{ is } r\text{-connected}.
\]
Definition

Let $[X, d, m]$ be a metric random walk with invariant measure $\nu$. $\Delta_m$ is said to be ergodic if $\Delta_m u = 0$ implies that $u$ is constant.
Infinite speed of propagation and $r$-connectedness

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Ergodicity implies that

\[ \lim_{t \to \infty} e^{t\Delta m} f = \int_X f(x) d\nu(x). \]
The \textit{m-total variation} of a function $u : X \to \mathbb{R}$ is defined as

$$TV_m(u) = \frac{1}{2} \int_X \int_X |u(y) - u(x)|dm_x(y)d\nu(x).$$

The \textit{m-perimeter} of a $\nu$-measurable subset $E \subset X$ is defined as

$$P_m(E) = TV_m(\chi_E) = \int_E \int_{X \setminus E} dm_x(y)d\nu(x).$$

where the last equality is consequence of the reversibility of $\nu$. 
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For $E \subset X$ $\nu$-mesurable and $x \in X$, we also define the \textit{mean m-curvature of} $\partial E$ \textit{at} $x$ as

$$\mathcal{H}_m^{\partial E}(x) = \int_{X} \left( \chi_{X \setminus E}(y) - \chi_E(y) \right) dm_x(y).$$
For $[\mathbb{R}^N, d, m^J]$ and $\nu = \mathcal{L}^N$, 

$$P_{m^J}(E) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\chi_E(y) - \chi_E(x)| J(x - y) dx,$$

and

$$\mathcal{H}_{\partial E}^{m^J}(x) = \int_{\mathbb{R}^N} J(x - y)(\chi_{\mathbb{R}^N \setminus E}(y) - \chi_E(y)) dy.$$

which coincides with the concepts we have studied in

[Mazón, Rossi, Toledo, *Nonlocal Perimeter, Curvature and Minimal Surfaces for measurable sets*, Journal d’Analyse Mathématique].
For weighted graphs, the definition of perimeter of a set $E \subset V(G)$ is given by

$$|\partial E| = \sum_{x \in E, y \in V \setminus E} w_{xy}.$$ 

Then we have that

$$|\partial E| = P_{mG}(E) \text{ for all } E \subset V(G).$$
Infinite speed of propagation and $r$-connectedness

Theorem

Let $[X, d, m]$ be a metric random walk with reversible measure $\nu$ and assume that $\nu(X) < +\infty$. Then

$\Delta_m$ is ergodic $\iff$ the $\nu$-mean value of the mean $m$-curvature of $\partial D$ in $D$ is

$$\frac{1}{\nu(D)} \int_D \mathcal{H}^m_{\partial D}(x) d\nu(x) > -1 \quad \forall D \text{ such that } 0 < \nu(D) < 1.$$
Let \([X, d, m]\) be a metric random walk space with reversible probability measure \(\nu\).

We denote the mean value of \(f \in L^1(X, \nu)\) (or the expected value of \(f\)) by

\[
\nu(f) = \mathbb{E}_\nu(f) = \int_X f(x) d\nu(x).
\]

And, for \(f \in L^2(X, \nu)\), we denote its variance by

\[
\text{Var}_\nu(f) = \int_X (f(x) - \nu(f))^2 d\nu(x) = \frac{1}{2} \int_{X \times X} (f(x) - f(y))^2 d\nu(y) d\nu(x).
\]
Let $[X, d, m]$ be a metric random walk space with reversible probability measure $\nu$.

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$$\nu(f) = \mathbb{E}_\nu(f) = \int_X f(x) d\nu(x).$$

And, for $f \in L^2(X, \nu)$, we denote its variance by

$$\text{Var}_\nu(f) = \int_X (f(x) - \nu(f))^2 d\nu(x) = \frac{1}{2} \int_{X \times X} (f(x) - f(y))^2 d\nu(y) d\nu(x).$$

**Definition**

The spectral gap of $-\Delta_m$ is defined as

$$\text{gap}(-\Delta_m) = \inf \left\{ \frac{\mathcal{H}_m(f)}{\text{Var}_\nu(f)} : f \in D(\mathcal{H}_m), \text{Var}_\nu(f) \neq 0 \right\}$$

$$= \inf \left\{ \frac{\mathcal{H}_m(f)}{\|f\|^2_2} : f \in D(\mathcal{H}_m), \|f\|^2_2 \neq 0, \int_X f d\nu = 0 \right\}.$$
Theorem

Let \([X, d, m]\) be a metric random walk space with reversible probability measure \(\nu\). Assume that \(\Delta_m\) is ergodic. Then

\[
gap(-\Delta_m) = \sup \left\{ \lambda \geq 0 : \lambda \mathcal{H}_m(f) \leq \int_X (-\Delta_m f)^2 \, d\nu \quad \forall f \in L^2(X, \nu) \right\}.
\]
We say that \((m, \nu)\) satisfies a Poincaré inequality if there exists \(\lambda > 0\) such that
\[
\lambda \text{Var}_\nu(f) \leq \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu),
\]
or equivalently,
\[
\lambda \|f\|_{L^2(X, \nu)}^2 \leq \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu) \text{ with } \nu(f) = 0.
\]
We say that \((m, \nu)\) satisfies a \textbf{Poincaré inequality} if there exists \(\lambda > 0\) such that

\[\lambda \text{Var}_\nu(f) \leq \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu),\]

or equivalently,

\[\lambda \|f\|_{L^2(X, \nu)}^2 \leq \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu) \text{ with } \nu(f) = 0.\]

Note that when \(\text{gap}(-\Delta_m) > 0\), \((m, \nu)\) satisfies a Poincaré inequality with \(\lambda = \text{gap}(-\Delta_m)\),

\[\text{gap}(-\Delta_m) \text{Var}_\nu(f) \leq \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu),\]

being the spectral gap the best constant in the Poincaré inequality.

A Poincaré inequality implies ergodicity.
The following statements are equivalent:

(i) There exists $\lambda > 0$ such that

$$\lambda \text{Var}_\nu(f) \leq \mathcal{H}_m(f)$$

for all $f \in L^2(X, \nu)$.

(ii) For every $f \in L^2(X, \nu)$

$$\|e^{t\Delta_m}f - \nu(f)\|_{L^2(X,\nu)} \leq e^{-\lambda t}\|f - \nu(f)\|_{L^2(X,\nu)}$$

for all $t \geq 0$. 

Theorem
For $G = (V(G), E(G))$ a finite connected weighted graph,

$$\text{gap}(-\Delta^G_m) > 0.$$ 

Let $J \in C(\mathbb{R}^N, [0, +\infty)$ be radially symmetric with $J(0) > 0$ and $\int J = 1$. For $\Omega \subset \mathbb{R}^N$ a bounded domain,

$$\text{gap}(-\Delta_{m,J,\Omega}) > 0.$$
For $G = (V(G), E(G))$ a finite connected weighted graph,

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$$\text{gap}(-\Delta_{mJ, \Omega}) > 0.$$ 

**Example**

Let $V(G) = \{x_3, x_4, x_5 \ldots, x_n \ldots\}$ be a weighted linear graph with

$$w_{x_{3n}, x_{3n+1}} = \frac{1}{n^3}, \quad w_{x_{3n+1}, x_{3n+2}} = \frac{1}{n^2}, \quad w_{x_{3n+2}, x_{3n+3}} = \frac{1}{n^3}.$$ 

$((m_x), \nu)$ does not satisfy a Poincaré inequality.
Functional inequalities and curvature

**Theorem (Ollivier)**

Let \([X, d, m]\) be a metric random walk space with reversible probability measure \(\nu\). Y. Ollivier, under the assumption that

\[
\int \int \int d(y, z)^2 dm_x(y) dm_x(z) d\nu(x) < +\infty,
\]

proves that if the Ollivier-Ricci curvature \(\kappa_m > 0\) then \((m, \nu)\) satisfies the Poincaré inequality

\[
\kappa_m \text{Var}_\nu(f) \leq \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu),
\]

and, consequently, \(\text{gap}(-\Delta_m) \geq \kappa_m > 0\).
Remark

The Poincaré inequality, tested only on characteristic functions, gives that there exists $\lambda > 0$ such that

$$\lambda \nu(D)(1 - \nu(D)) \leq P_m(D) \quad \text{for all } \nu-\text{mesasurable sets } D,$$

which implies the following isoperimetric inequality:

$$\min\{\nu(D), 1 - \nu(D)\} \leq \frac{2}{\lambda} P_m(D);$$
In a weighted graph $G = (V(G), E(G))$ the Cheeger constant is defined as

$$h_G = \inf_{D \subset V(G)} \frac{|\partial D|}{\min\{\nu_G(D), \nu_G(V(G) \setminus D)\}}.$$ 

The following relation between the Cheeger constant and the first positive eigenvalue $\lambda_1(G)$ of the graph Laplacian $\Delta_{m_G}$ is well-known:

$$\frac{h_G^2}{2} \leq \lambda_1(G) \leq 2h_G.$$
Let $[X, d, m]$ be a metric random walk space with reversible probability measure $\nu$, its Cheeger constant is defined as

$$h_m(X) = \inf \left\{ \frac{P_m(D)}{\min\{\nu(D), \nu(X \setminus D)\}} : D \subset X, \ 0 < \nu(D) < 1 \right\},$$
Let $[X, d, m]$ be a metric random walk space with reversible probability measure $\nu$, its **Cheeger constant** is defined as

$$
\text{h}_m(X) = \inf \left\{ \frac{P_m(D)}{\min\{\nu(D), \nu(X \setminus D)\}} : D \subset X, \ 0 < \nu(D) < 1 \right\},
$$

Given a function $u : X \to \mathbb{R}$, we say that $\mu \in \mathbb{R}$ is a **median** of $u$ with respect to a measure $\nu$ ($\mu \in \text{med}_\nu(u)$) if

$$
\nu(\{x \in X : u(x) < \mu\}) \leq \frac{1}{2} \nu(X) \ \& \ \nu(\{x \in X : u(x) > \mu\}) \leq \frac{1}{2} \nu(X).
$$
Let $[X, d, m]$ be a metric random walk space with reversible probability measure $\nu$, its **Cheeger constant** is defined as

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$$\nu(\{x \in X : u(x) < \mu\}) \leq \frac{1}{2}\nu(X) \ \& \ \nu(\{x \in X : u(x) > \mu\}) \leq \frac{1}{2}\nu(X).$$

**Theorem**

*If $[X, d, m]$ is a metric random walk space with reversible probability measure $\nu$, then*

$$h_m(X) = \inf \{ TV_m(u) : \|u\|_1 = 1, \ 0 \in \text{med}_\nu(u) \}. $$
Following F. Chung, Spectral Graph Theory, 1997,

**Theorem**

Let \([X, d, m]\) be a metric random walk space with reversible probability measure \(\nu\). The following Cheeger inequality holds

\[
\frac{h_m^2}{2} \leq \text{gap}(-\Delta_m) \leq 2h_m.
\]
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**Theorem**

*Let* \([X, d, m]\) *be a metric random walk space with reversible probability measure* \(\nu\). *The following Cheeger inequality holds*

\[
\frac{h_m^2}{2} \leq \text{gap}(-\Delta_m) \leq 2h_m.
\]

**Corollary**

The following statements are equivalent:

1. \((m, \nu)\) satisfies a Poincaré inequality,
2. \(\text{gap}(-\Delta_m) > 0\),
3. \((m, \nu)\) satisfies an isoperimetric inequality,
4. \(h_m(X) > 0\).
We can study the Bakry-Émery curvature condition in this context since we have a Carré du champ $\Gamma$ defined by

$$\Gamma(f, g)(x) = \frac{1}{2} \left( \Delta_m (fg)(x) - f(x) \Delta_m g(x) - g(x) \Delta_m f(x) \right)$$

for $f, g \in L^2(X, \nu)$.

$$\Gamma(f, g)(x) = \frac{1}{2} \int_X \nabla f(x, y) \nabla g(x, y) dm_x(y).$$
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for $f, g \in L^2(X, \nu)$.

According to Bakry and Émery, we define the Ricci curvature operator $\Gamma_2$ by iterating $\Gamma$:

$$\Gamma_2(f, g) = \frac{1}{2} \left( \Delta_m \Gamma(f, g) - \Gamma(f, \Delta_m g) - \Gamma(\Delta_m f, g) \right),$$

which is well defined for $f, g \in L^2(X, \nu)$. 

Julián Toledo

**The Heat Flow on Metric Random Walk Spaces**
Bakry-Émery curvature condition

We can study the Bakry-Émery curvature condition in this context since we have a Carré du champ $\Gamma$ defined by

$$\Gamma(f, g)(x) = \frac{1}{2} \left( \Delta_m (fg)(x) - f(x) \Delta_m g(x) - g(x) \Delta_m f(x) \right)$$ for $f, g \in L^2(X, \nu)$.

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According to Bakry and Émery, we define the Ricci curvature operator $\Gamma_2$ by iterating $\Gamma$:

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which is well defined for $f, g \in L^2(X, \nu)$.

We write, for $f \in L^2(X, \nu)$,

$$\Gamma(f) = \Gamma(f, f) \quad \text{and} \quad \Gamma_2(f) = \Gamma_2(f, f).$$

We have that

$$\int_X \Gamma(f) d\nu = \mathcal{H}_m(f) \quad \text{and} \quad \int_X \Gamma_2(f) d\nu = \int_X (\Delta_m f)^2 d\nu.$$
Bakry-Émery curvature condition

Definition

The operator \( \Delta_m \) satisfies a Bakry-Émery curvature condition \( BE(K, n) \) for \( n \in (1, +\infty) \) and \( K \in \mathbb{R} \) if

\[
\Gamma_2(f) \geq \frac{1}{n} (\Delta_m f)^2 + K \Gamma(f) \quad \forall \, f \in L^2(X, \nu).
\]

The constant \( n \) is called the dimension, and \( K \) is called a lower bound of the Ricci curvature of \( \Delta_m \).
The operator $\Delta_m$ satisfies a Bakry-Émery curvature condition $BE(K, n)$ for $n \in (1, +\infty)$ and $K \in \mathbb{R}$ if

$$\Gamma_2(f) \geq \frac{1}{n}(\Delta_m f)^2 + K\Gamma(f) \quad \forall f \in L^2(X, \nu).$$

The constant $n$ is called the dimension, and $K$ is called a lower bound of the Ricci curvature of $\Delta_m$.

If there exists $K \in \mathbb{R}$ such that

$$\Gamma_2(f) \geq K\Gamma(f) \quad \forall f \in L^2(X, \nu),$$

then it is said that the operator $\Delta_m$ satisfies a Bakry-Émery curvature condition $BE(K, \infty)$. 
Integrating the Bakry-Émery curvature condition we get:

**Theorem**

Let $[X, d, m]$ be a metric random walk with reversible probability measure $\nu$. Then, if $\Delta_m$ satisfies a Bakry-Émery curvature condition $BE(K, n)$ with $K > 0$, we have

$$\text{gap}(-\Delta_m) \geq K \frac{n}{n-1}.$$  

In the case that $\Delta_m$ satisfies a Bakry-Émery curvature condition $BE(K, \infty)$ with $K > 0$, we have

$$\text{gap}(-\Delta_m) \geq K.$$
Example

Consider the non weighted linear graph $G$ with vertices $V(G) = \{a, b, c\}$ (that is, the non-null weights are $w_{a,b} = w_{b,c} = 1$). We have that its graph Laplacian satisfies

$$BE \left( \frac{n-2}{n}, n \right) \quad \text{for any} \quad n > 1,$$

being $K = \frac{n-2}{n}$ the best constant for a fixed $n > 1$. And

$$\text{gap}(-\Delta) = 1.$$
Let \([X, d, m]\) be a metric random walk space with reversible probability measure \(\nu\).

For \(\mu \ll \nu\) with \(\frac{d\mu}{d\nu} = f\), we will write \(\mu = f\nu\).

The **Fisher-Donsker-Varadhan information** of a probability measure \(\mu\) on \(X\) with respect to \(\nu\) is defined by

\[
I_\nu(\mu) = \begin{cases} 
2\mathcal{H}_m(\sqrt{f}) & \text{if } \mu = f\nu, ~ f \geq 0, \\
\infty & \text{otherwise.}
\end{cases}
\]
Let $[X, d, m]$ be a metric random walk space with reversible probability measure $\nu$. For $x \in X$ we define

$$\Theta(x) = \frac{1}{2} \int_X d(x, y)^2 dm_x(y),$$

and

$$\Theta_m = \sup_{x \in X} \Theta(x).$$

**Theorem**

Assume that $\Theta_m$ is finite. If $\Delta_m$ satisfies the Bakry-Émery curvature condition $BE(K, \infty)$ with $K > 0$, then $\nu$ satisfies the transport-information inequality

$$W_1^d(\mu, \nu) \leq \frac{\sqrt{\Theta_m}}{K} \sqrt{I_\nu(\mu)}$$

for all probability measure $\mu \ll \nu$. 

The Heat Flow on Metric Random Walk Spaces
Theorem

Let \([X, d, m]\) be a metric random walk space with reversible probability measure \(\nu\). Assume that \(\Theta_m\) is finite. If the Ollivier-Ricci curvature \(\kappa_m > 0\), then the following transport-information inequality holds:

\[
W_1^d(\mu, \nu) \leq \frac{\sqrt{2\Theta_m}}{\kappa_m} \sqrt{I_{\nu}(\mu)} \quad \text{for all probability measure } \mu \ll \nu.
\]
The $m$-1-Laplacian

Let $[X, d, m]$ be a metric random walk space with reversible measure $\nu$. And let $\mathcal{F}(u) : L^2(\Omega, \nu) \to [0, +\infty]$ defined as

$$\mathcal{F}(u) = TV_m(u) = \frac{1}{2} \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x)$$

when finite. It is convex and lower semicontinuous. $\partial \mathcal{F}_m$ is $m$-completely accretive. Then we can obtain a strong solution of

$$\begin{cases}
\frac{du}{dt} - \Delta_1^m u(t) \geq 0 & \text{in } (0, T) \times X, \\
\ u(0) = u_0.
\end{cases}$$
Let \([X, d, m]\) be a metric random walk space with reversible measure \(\nu\). And let \(\mathcal{F}(u) : L^2(\Omega, \nu) \to [0, +\infty]\) defined as

\[
\mathcal{F}(u) = TV_m(u) = \frac{1}{2} \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x)
\]

when finite. It is convex and lower semicontinuous. \(\partial \mathcal{F}_m\) is \(m\)-completely accretive. Then we can obtain a strong solution of

\[
\begin{cases}
    \frac{du}{dt} - \Delta_1^m u(t) \ni 0 & \text{in } (0, T) \times X, \\
    u(0) = u_0.
\end{cases}
\]

\(\nu \in \partial \mathcal{F}_m(u) \iff\) there exists \(g \in L^\infty(X \times X, \nu \otimes m_x)\) antisymmetric with \(\|g\|_\infty \leq 1\) such that

\[-\int_X g(x, y) dm_x(y) = \nu(x) \quad \nu - \text{a.e } x \in X,
\]

and

\(g(x, y) \in \text{sign}(u(y) - u(x)) \quad (\nu \otimes m_x) - \text{a.e. } (x, y) \in X \times X.\)
Merci de Votre Attention