A Lagrangian scheme "à la Brenier" for the incompressible Euler equations

T. O. Gallouët[1] , Q. Mérigot [2]

Inria MOKAPLAN [1]
Université Paris Sud-Orsay [2]

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Incompressible Euler equations

Domain: $\Omega \subseteq \mathbb{R}^d$ with Leb measure. Eulerian formulation:

$$\begin{cases} \partial_t u(t,x) + (u(t,x)\cdot\nabla)\, u(t,x) = -\nabla p(t,x), \\ \operatorname{div}(u(t,x)) = 0 \\ u(t,x)\cdot n = 0 \\ u(0,x) = u_0. \end{cases}$$

Lagrangian formulation:

$$\begin{cases} \frac{d}{dt}\phi(t,x) = u\,(t,\phi(t,x)) \quad \text{for } t\in[0,T],\ x\in\Omega\,, \\ \phi(0,\cdot) = \mathrm{id}, \\ \partial_t\phi(0,\cdot) = u_0. \end{cases}$$

Incompressible Euler equations II

Measure preserving maps:

$$\mathbb{S} = \left\{ s \in \mathrm{L}^2(\Omega, \mathbb{R}^d) \mid s_\# \,\mathrm{Leb} = \mathrm{Leb} \right\},$$

The incompressibility constraint reads

$$\mathrm{div}\, u = 0 \iff \det \nabla\phi(t) = 1 \iff \phi(t,\cdot) \in \mathbb{S}$$

The evolution equation

$$\frac{d^2}{dt^2}\phi(t) = -\nabla p(t, \phi(t,x)) \in (T_\phi \mathbb{S})^\perp$$

Conclusion: solutions of the incompressible Euler equations are geodesics of the measure preserving maps $\mathbb{S}$ for the L^2 metric.

Approached geodesics à la Brenier

Simple example: we consider $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$.

geodesic: γ: $[0, T] \to \mathbb{R}^2$ with

$$\begin{cases} \gamma(t) = (t, 0),\ t \in [0, T], \\ \gamma(0) = (0, 0), \\ \dot{\gamma}(0) = (1, 0). \end{cases}$$

Approached geodesic z with initial error δ.

$$\begin{cases} z(0) = (0, \delta), \\ \dot{z}(0) = (1, 0). \end{cases}$$

ϵ-evolution :

$$\ddot{z}(t) = \frac{1}{\epsilon^2} \left(\mathbb{P}_{\mathbb{R}}(z) - z \right)$$

Approached geodesics à la Brenier II

Exact solution:

$$z(t) = \left(t, \delta \cos \frac{t}{\epsilon}\right).$$

Solution for the Hamiltonian system defined with

$$H(z, v) = \frac{1}{2}||v||^2 + \frac{1}{2\epsilon^2} d^2_{\mathbb{R}\times\{0\}}(z)$$

Quantifying the convergence: the modulated energy

$$E_\gamma(t) = \frac{1}{2}||\dot{z}(t) - \dot{\gamma}(t)||^2 + \frac{1}{2\epsilon^2} d^2_{\mathbb{R}\times\{0\}}(z(t)).$$

$$E_\gamma(t) = \frac{1}{2}\frac{\delta^2}{\epsilon^2}$$

naive idea

Compute the solution of the Hamiltonian system associated to

$$H(f, v) = \frac{1}{2}\|v\|_{\mathbb{M}}^2 + \frac{d_{\mathbb{S}}^2(f)}{2\epsilon^2}.$$

How to compute $d_{\mathbb{S}}^2(f)$?

$\rightarrow$ Optimal Transport (Brenier)

Settings

- two probability measures μ, ν on $\mathbb{R}^d$.
- cost $c : X \times Y \to \mathbb{R}$, in particular $c(x, y) = |x - y|^2$.
- $\mathbf{t}_{\#}\mu = \nu$: for any measurable set B we have $\nu(B) = \mu\left(\mathbf{t}^{-1}(B)\right)$.

Problem (Monge 1781)

Find T such that

$$T = \underset{\mathbf{t}_{\#}\mu=\nu}{\operatorname{argmin}} \left(\int c\left(x, \mathbf{t}(x)\right) d\mu(x) \right).$$

$$\mathtt{MK}_2^2(\mu, \nu) = \min_{\mathbf{t}_{\#}\mu=\nu} \left(\int \left|\mathbf{t}(x) - x\right|^2 d\mu(x) \right)$$

Existence

Theorem (Existence, Brenier)

Let μ ac. w.r.t. Leb. *Then there exists a unique $\nabla\varphi$ with φ convex such that $T = \nabla\varphi$.*

Theorem (Polar decomposition, Brenier)

Let $f \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$ and $\nu = f_\#$ Leb. Then there exists φ convex and $\sigma \in \mathbb{S}$ such that

- $f = \nabla\varphi \circ \sigma$,
- $d_{\mathbb{S}}^2(f) = \mathtt{MK}_2^2(\mathrm{Leb}, \nu) = \mathtt{MK}_2^2(\mathrm{Leb}, f_\#\,\mathrm{Leb})$,
- $\sigma \in P_{\mathbb{S}}(f)$.

Kantorovich potentials

Owner point of view:

Factory points: X_α, $\alpha \in \{1, ..., N\}$, producing $\frac{1}{N}$.
Distribution points: Y_β, $\beta \in \{1, ..., N\}$, distributing $\frac{1}{N}$.
Transport cost: $c(X_\alpha, Y_\beta)$.

Shipper point of view: At a factory X_α, I buy $-\varphi(X_\alpha)$. At a distribution point Y_β, I sell $\psi(Y_\beta)$. I must ship everything and I guarantee for all α, β

$$(**)\ \varphi(X_\alpha) + \psi(Y_\beta) \leq c(X_\alpha, Y_\beta).$$

Kantorovich potentials II

The $(**)$ deal means

$$\int \varphi(x) d\mu(x) + \int \psi(y) d\nu(y) \leq \mathtt{MK}_2^2(\mu, \nu).$$

Kantorovich duality

$$\sup_{\varphi \bigoplus \psi \leq c} \int \varphi(x) d\mu(x) + \int \psi(y) d\nu(y) = \mathtt{MK}_2^2(\mu, \nu).$$

c-transform

$$\psi^c(x) = \sup_y \left(c(x, y) - (y) \right).$$

Semi-Discrete optimal transport (Q. Mérigot)

We want to compute $\mathtt{MK}_2^2\left(\text{Leb}, \frac{1}{N}\sum_{\beta=1}^{N}\delta_{Y_\beta}\right)$.

Laguerre Cells for points Y_β and prices ψ_β.

$$\mathrm{Lag}_\alpha\left((Y_\beta)_{1\dots N}, (\psi_\beta)_{1\dots N}\right) = \\ \left\{x \in \Omega \middle| -\psi_\alpha + |x - Y_\alpha|^2 \leq -\psi_\beta + |x - Y_\beta|^2\,,\ \text{for all } \beta\right\}$$

$\rightarrow$ Partition of the space Ω.

To be sure that the price gives a transport maps, it must holds

$$|\mathrm{Lag}_\alpha\left(Y_\beta, \psi_\beta\right)| = 1/N.$$

Numerically: Newton on this condition.

Discretization space

Ω partitioned in N cells P_α with mass $\frac{1}{N}$.

$$\mathbb{M}_N = \{ \text{ piecewise constant functions on } P_\alpha \} \subset L^2(\Omega, \mathbb{R}^d).$$

For $f \in \mathbb{M}_N$:

$$d^2_{\mathbb{S}}(f) = \mathtt{MK}^2_2(\text{Leb}, f_\# \text{Leb}) = \mathtt{MK}^2_2(\text{Leb}, \frac{1}{N}\sum_{\alpha=1}^{N} \delta_{f_\alpha}).$$

Projection set of f on $\mathbb{S}$:

$$\mathbb{P}_{\mathbb{S}}(f) = \{\sigma \in \mathbb{S} \mid \forall \alpha, \sigma(P_\alpha) = \text{Lag}_\alpha(f_\beta, \psi^{opt})\}.$$

Approximate Geodesics

$d_{\mathbb{S}}^2$ as a function : $\mathbb{M}_N \to \mathbb{R}$:

$$H(f, v) = \frac{1}{2}\|v\|_{\mathbb{M}}^2 + \frac{d_{\mathbb{S}}^2(f)}{2\epsilon^2}$$
$$= \frac{1}{2}\|v\|_{\mathbb{M}}^2 + \frac{1}{2\epsilon^2} MK_2^2(\text{Leb}, \frac{1}{N}\sum_{\alpha=1}^{N}\delta_{f_\alpha}).$$

Scheme:

$$\begin{cases} \ddot{f}(t) + \frac{\nabla d_{\mathbb{S}}^2(f(t))}{2\epsilon^2} = 0, & \text{for } t \in [0, T], \\ (f(0), \dot{f}(0)) \in \mathbb{M}_N^2 \end{cases}$$

With

$$\frac{1}{2}\nabla d_{\mathbb{S}}^2(f(t)) = f - \mathbb{P}_{\mathbb{M}} \circ \mathbb{P}_{\mathbb{S}}(f) \in \mathbb{M}_N$$

Approximate Geodesics II

$\mathbb{P}_{\mathbb{M}} \circ \mathbb{P}_{\mathbb{S}}(f)$ is the application

$$P_\alpha \to \mathrm{Bar}\left(\mathrm{Lag}_\alpha(f_\beta, \psi^{opt})\right)$$

Scheme rewrites

$$\begin{cases} \ddot{f}(t) = \frac{\mathbb{P}_{\mathbb{M}} \circ \mathbb{P}_{\mathbb{S}}(f) - f}{\epsilon^2} = 0, & \text{for } t \in [0, T], \\ (f(0), \dot{f}(0)) \in \mathbb{M}_N^2 \end{cases}$$

Full discretization (Euler semi-symplectic), step time τ.

$$\begin{cases} V^{n+1} & = V^n - \tau \frac{f^n - P_{\mathbb{M}_N} \circ P_{\mathbb{S}}(f^n)}{\epsilon^2} \\ f^{n+1} & = f^n + \tau V^{n+1}, \end{cases}$$

Convergence

Let u be a smooth (eulerian) solution of the incompressible Euler equations.

The modulated energy:

$$E_{u(t)} = \frac{1}{2}\|\dot{f}(t) - u(t, f(t))\|^2_{\mathbb{M}} + \frac{d^2_{\mathbb{S}}(f)}{2\epsilon^2}.$$

Grönwall estimates, Semi-Discrete:

$$E_{u(t)} \leq C\left[\epsilon^2 + h_N + \frac{h_N^2}{\epsilon^2}\right]$$

Full discrete:

$$E_{u(t)} \leq C\left[\epsilon^2 + h_N + \frac{h_N^2}{\epsilon^2} + \kappa + \frac{\tau}{\epsilon}\right]$$

Where

$$\kappa = \max_{n \in \mathbb{N} \cap [0, T/\tau]} \left(H^n - H^0\right) \leq \frac{\tau}{\epsilon^2}.$$

Fluid-structure interactions

Structure domain : $t \mapsto \mathcal{S}(t)$ with constant volume.

Fluid domain : $\mathcal{F}(t) = \Omega \setminus \overline{\mathcal{S}(t)}$

Hamiltonian : $H(f, v) = \frac{1}{2}\|v\|^2_{\mathbb{M}} + \frac{d^2_{\mathbb{S}_t}(f)}{2\epsilon^2}$

where $\mathbb{S}_t = \left\{ s \in \mathrm{L}^2(\mathcal{F}(0), \mathbb{R}^d) \mid s_{\#}\mathrm{Leb}_{\mathcal{F}(0)} = \mathrm{Leb}_{\mathcal{F}(t)} \right\}$.

- Convergence of the scheme, presence of additional terms.
- Still some problems when the structure is not known.

The Riemannian submersion

Space: $(\mathbb{S}, \mathrm{L}_2) \subset L^2(\Omega)$

Projection: $d_{\mathbb{S}}^2(f(t)) = \mathtt{MK}_2^2(\mathrm{Leb}, f_\# \mathrm{Leb})$, polar decomposition.

Geodesics: Incompressible Euler equations.

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Behind the structure: the Otto Riemannian submersion

$$\pi : (\mathsf{Diff}(\Omega), \mathrm{L}_2) \to (\mathsf{Dens}(\Omega), \mathtt{MK}_2)$$
$$\phi \mapsto \phi_\#(\mathsf{Leb}).$$

Definition

The map π is a Riemannian submersion if π is a submersion and for any $x \in M$, the map $df_x : \mathrm{Ker}(d\pi_x)^\perp \mapsto T_{\pi(x)}N$ is an isometry.

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$$v = u + \nabla p, \quad \mathsf{div}(u) = 0, \quad \mathtt{W}_2^2(\partial_t \rho) = \inf_{\partial_t \rho = \mathsf{div}(\rho v)} \|u\|_{\mathrm{L}^2(\rho)} = \|\nabla p\|_{\mathrm{L}^2(\rho)}$$

The Riemannian submersion

What if

Projection: given by the `WMKHFR` distance on $\mathcal{M}_+(\Omega)$

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Submersion

$$\pi : \big(\mathrm{Diff}(\Omega) \ltimes C^\infty(\Omega, \mathbb{R}^+), d_c\big) \to \quad (\mathcal{M}_+(\Omega), \mathtt{WFR})$$
$$(\phi, \lambda) \mapsto \pi(\phi, \lambda) = \phi_\#(\lambda^2 \, \mathrm{Leb}).$$

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$$\mathtt{WFR}_2^2(\partial_t \rho) = \inf_{\partial_t \rho = \mathrm{div}(\rho v) + r\rho} \|v\|^2_{\mathrm{L}^2(\rho)} + \|r\|^2_{\mathrm{L}^2(\rho)} = g_c(\nabla r, r)$$

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Space: $\tilde{\mathbb{S}} \subset \mathrm{Diff}(\Omega) \ltimes C^\infty(\Omega, \mathbb{R}^+)$

Projection: given by the HK-WFR distance on $\mathcal{M}_+(\Omega)$ New polar decomposition

Geodesics:

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Geodesics: Camassa-Holm equations.

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Conclusion

Revisit Brenier's work for the incompressible Euler equations in this case

- Generalized solutions
- Regularity of the pression
- Numerical scheme

New polar decompostion:

$$(\phi, \lambda) = \exp^{\mathcal{C}(M)}\left(-\frac{1}{2}\nabla p_{z_0}, -p_{z_0}\right) \circ (s, \sqrt{\mathrm{Jac(s)}})$$

or equivalently

$$(\phi, \lambda) = \left(\varphi, e^{-z_0}\sqrt{1 + \|\nabla z_0\|^2}\right) \cdot (s, \sqrt{\mathrm{Jac(s)}})\,,$$

where $p_{z_0} = e^{z_0} - 1$ and

$$\varphi(x) = \exp_x^M\left(-\operatorname{arctan}\left(\frac{1}{2}\|\nabla z_0(x)\|\right)\frac{\nabla z_0(x)}{\|\nabla z_0(x)\|}\right)\,.$$