On some approximation results in Musielak spaces

Ahmed YOUSSFI

Collaborators : Y. Ahmida (ENSA, Fez)
I. Chlebicka, P. Gwiazda (Polish Academy of Sciences, Warsaw)

National School of Applied Sciences
University Sidi Mohamed Ben Abdellah–Fez, Morocco

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Flow of non-Newtonian fluid

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$$
\begin{aligned}
\partial_t v + (u \cdot \nabla) v - \operatorname{div} S(t, x, v, Dv) + \nabla \pi &= f && \text{in } Q \\
\operatorname{div} v &= 0 && \text{in } Q \\
v(0, x) &= v_0 && \text{in } \Omega \\
v(t, x) &= 0 && \text{on } (0, T) \times \partial\Omega
\end{aligned}
$$

where
$v : Q \to \mathbb{R}^N$ denotes the velocity field,
$\pi : Q \to \mathbb{R}$ the pressure,
S the stress tensor,
$f : Q \to \mathbb{R}^N$,
Ω is a bounded domain with a smooth boundary $\partial\Omega$
$Q = (0, T) \times \Omega$ with some given $T > 0$.
$Dv = \frac{1}{2}(\nabla v + \nabla^T v)$

Part I

Some approximation results in Musielak spaces

Φ-functions

Let Ω be an open of $\mathbb{R}^N, N \geq 1$. A real function $M : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a Φ-function, $(M \in \Phi)$, if

1. $M(x, \cdot)$ is a convex function for all $x \in \Omega$.
2. $M(x, 0) = 0$, $M(x, s) > 0$ for $s > 0$, $M(x, s) \to \infty$ as $s \to \infty$.
3. $M(\cdot, s)$ is a measurable function for every $s \geq 0$.
4. $M(x, s)/s \to 0$ as $s \to 0^+$ and $M(x, s)/s \to \infty$ as $s \to \infty$.

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Its complementary Φ-function

$$M^*(x, s) = \sup_{t \geq 0}\{st - M(x, t)\} \text{ for all } s \geq 0 \text{ and all } x \in \Omega.$$

Musielak spaces

$$L_M(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega M(x, |u(x)|/\lambda)dx < +\infty \text{ for some } \lambda > 0 \right\},$$

$$E_M(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega M(x, |u(x)|/\lambda)dx < +\infty \text{ for all } \lambda > 0 \right\}.$$

The Δ_2-condition

We say that $M \in \Phi$ satisfies the Δ_2-condition, written $M \in \Delta_2$, if there exist a constant $k > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$M(x, 2t) \leq kM(x, t) + h(x), \quad (1)$$

for all $t \geq 0$ and for almost every $x \in \Omega$.

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Remark 1

$M \in \Delta_2 \Longleftrightarrow L_M(\Omega) = E_M(\Omega)$.

Modular convergence

A sequence $\{u_k\}_k$ is said to converge modularly to u in $L_M(\Omega)$ if there exists $\lambda > 0$ such that

$$\rho_M((u_k - u)/\lambda) := \int_\Omega M\left(x, |u_k - u|/\lambda\right) dx \to 0 \text{ as } k \to \infty.$$

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$$M(x,s) = e^{s^2} - 1; \quad M(x,s) = w(x)(\exp(s) - 1 + s), s \geq 0.$$

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$$M(x,s) = e^{s^2} - 1; \quad M(x,s) = w(x)(\exp(s) - 1 + s), s \geq 0.$$

6. Poincaré-type inequalities fail to hold in Musielak-Sobolev spaces.

Local integrability condition

Local integrability

A Φ-function M is locally integrable if for any constant number $c > 0$ and for any compact set $K \subset \Omega$ it holds

$$\int_K M(x,c)dx < \infty. \qquad (\mathcal{L}^1_{loc})$$

Local integrability condition

$(\mathcal{L}^1_{loc})$ is not always satisfied.

Example. Set $\Omega = (-1/2, 1/2)$ and set

$$M(x,s) = \begin{cases} s^{1/x} & x \in (0, 1/2), \\ s^2 & x \in (-1/2, 0). \end{cases}$$

Note that M is a Φ-function. Consider the compact set $K = [0, 1/4] \subset \Omega$. Then, for any $c > 1$

$$\int_K M(x,c)dx = \int_0^{1/4} c^{1/x} dx = +\infty.$$

Local integrability condition

Examples of $M \in \Phi$ for which $(\mathcal{L}^1_{loc})$ is filled (when Ω is of finite Lebesgue measure).

- $M(x,s) = M(s)$ (Orlicz spaces).
- $M(x,s) = s^{p(x)}$, $1 < p^- \leq p(x) \leq p^+ < +\infty$.
- $M(x,s) = |s|^p + a(x)|s|^q$, $a(\cdot) \in L^1_{loc}(\Omega)$.

Consequences of $\mathcal{L}^1_{loc}$

Let

- $\mathcal{B}_c(\Omega)$ be the set of bounded functions compactly supported in Ω.

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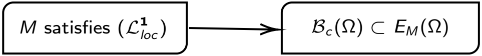

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One has

$$M \text{ satisfies } (\mathcal{L}^1_{loc}) \longrightarrow \mathcal{B}_c(\Omega) \subset E_M(\Omega) \longrightarrow \mathcal{C}_0^\infty(\Omega) \subset E_M(\Omega)$$

Shift operator

$$\tau_h u(x) = \begin{cases} u(x+h) & \text{if } x \in \Omega \text{ and } x+h \in \Omega, \\ 0 & \text{otherwise in } \mathbb{R}^N. \end{cases}$$

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Example : Kováčik-Rákosník, 1991.
$M(x,t) = t^{p(x)}$ $N = 1$, $\Omega = (-1,1)$. For $1 \leq r < s < +\infty$ they define

$$p(x) = \begin{cases} r & \text{if } x \in [0,1), \\ s & \text{if } x \in (-1,0) \end{cases} , \quad f(x) = \begin{cases} x^{-\frac{1}{s}} & \text{if } x \in [0,1), \\ 0 & \text{if } x \in (-1,0). \end{cases}$$

They show that $\tau_h f \notin L^{p(\cdot)}(\Omega)$ although that $f \in L^{p(\cdot)}(\Omega)$. Observe here, in this example, that the function f is compactly supported but not bounded on Ω.

Shift operator : M-mean continuity

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Theorem 1 (M-mean continuity)

Let $M \in \Phi$ satisfy $(\mathcal{L}^1_{loc})$. Then, any $u \in \mathcal{B}_c(\Omega)$ is M-mean continuous, that is to say for every $\varepsilon > 0$ there exists a $\eta = \eta(\varepsilon) > 0$ such that for $h \in \mathbb{R}^N$ with $|h| < \eta$ we have

$$\|\tau_h u - u\|_M < \varepsilon.$$

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$$\|\tau_h u - u\|_M < \varepsilon.$$

N.B. : The boundedness of the function u is necessary, otherwise the result in Theorem 1 is false in general.

Auxiliary results

Denote by J the Friedrichs mollifier kernel defined on $\mathbb{R}^N$ by

$$J(x) = ke^{-\frac{1}{1-\|x\|^2}} \text{ if } \|x\| < 1 \text{ and } 0 \text{ if } \|x\| \geq 1,$$

where $k > 0$ is such that $\int_{\mathbb{R}^N} J(x)dx = 1$. For $\varepsilon > 0$, we define

$$u_\varepsilon(x) = J_\varepsilon * u(x) = \int_{\mathbb{R}^N} J_\varepsilon(x-y)u(y)dy = \int_{B(0,1)} u(x-\varepsilon y)J(y)dy.$$

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A direct consequence of Theorem 1 is the following.

Corollary 2 (Norm convergence of approximate identities)

Let M be a Φ-function satisfying $(\mathcal{L}^1_{loc})$ and let $\underline{u \in \mathcal{B}_c(\Omega)}$. For any $\varepsilon > 0$ small enough, we have $u_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$. Furthermore,

$$\|u_\varepsilon - u\|_{L_M(\Omega)} \to 0 \text{ as } \varepsilon \to 0^+.$$

Auxiliary results

Theorem 3

Let M be a Φ-function satisfying $(\mathcal{L}^1_{loc})$. Then

1. $\mathcal{B}_c(\Omega)$ *is dense in* $E_M(\Omega)$ *with respect to the strong topology in* $L_M(\Omega)$.
2. $\mathcal{B}_c(\Omega)$ *is dense in* $L_M(\Omega)$ *with respect to the modular topology in* $L_M(\Omega)$.

Basic density results in Musielak spaces

Theorem 4

Let M be a Φ-function satisfying $(\mathcal{L}^1_{loc})$. Then,

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For any $M \in \Phi$ satisfying $(\mathcal{L}^1_{loc})$, the space $E_M(\Omega)$ is separable.

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Theorem 6 (Duality)

Let M be a Φ-function satisfying $(\mathcal{L}^1_{loc})$ and let M^* stands for its complementary function. Then, the dual space $(E_M(\Omega))'$ of $E_M(\Omega)$ is isomorphic to $L_{M^*}(\Omega)$, denote $(E_M(\Omega))' \simeq L_{M^*}(\Omega)$.

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Theorem 7 (Reflexivity)

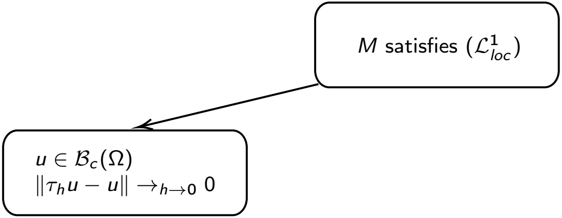
Let $M, M^* \in \Phi$ be a pair of complementary Φ-functions satisfying both $(\mathcal{L}^1_{loc})$ and the Δ_2-condition. Then, the Musielak-Orlicz space $L_M(\Omega)$ is reflexive.

Summarizing

M satisfies $(\mathcal{L}^1_{loc})$

Figure: Gathered results

Summarizing



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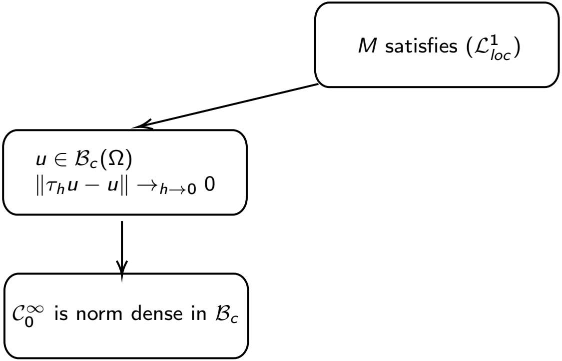

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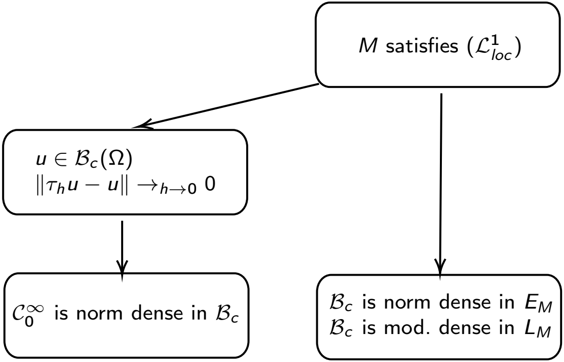

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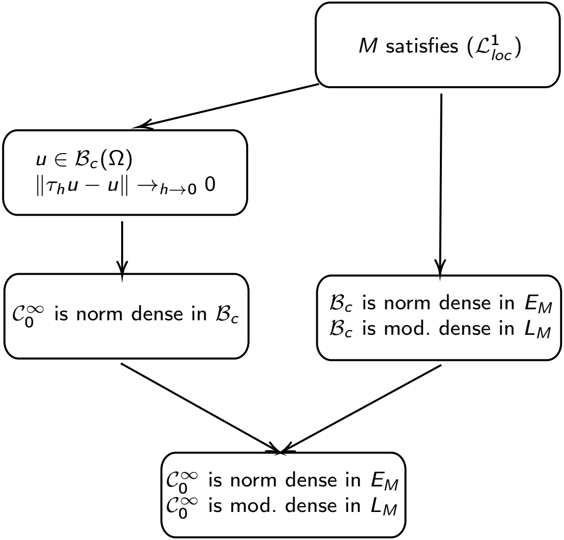

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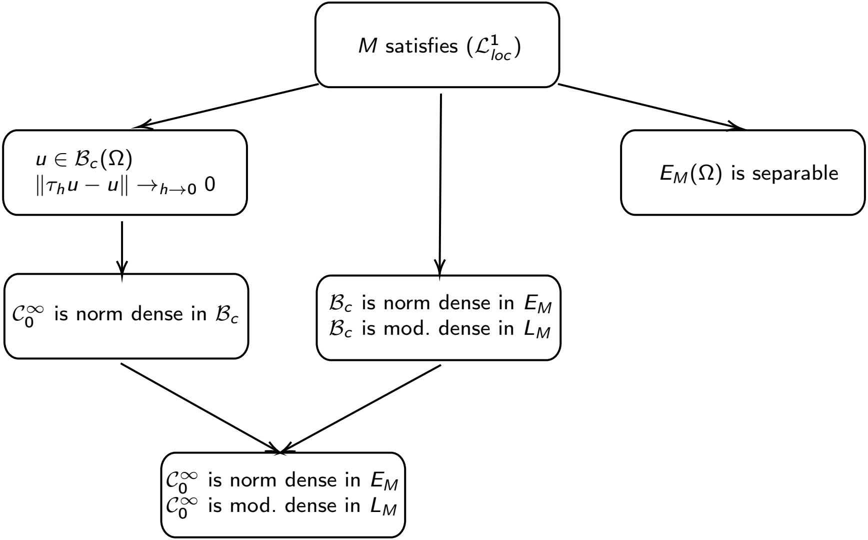

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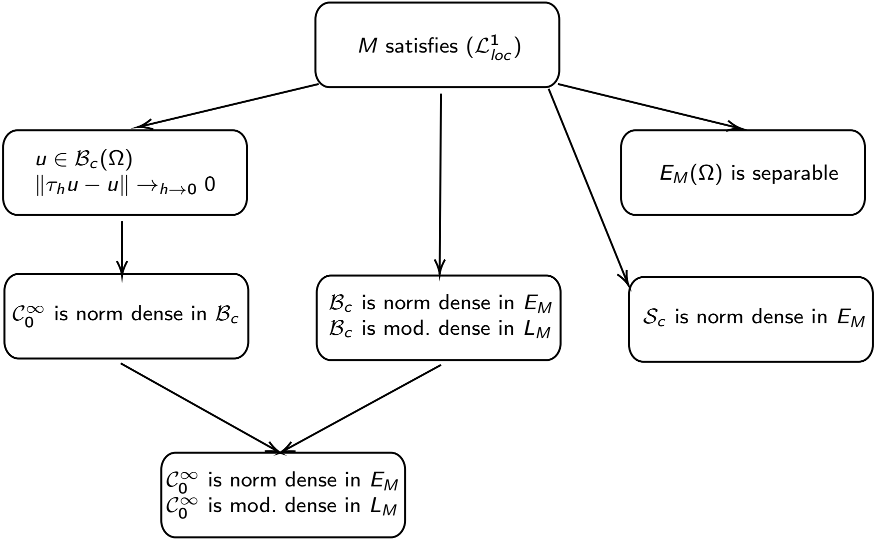

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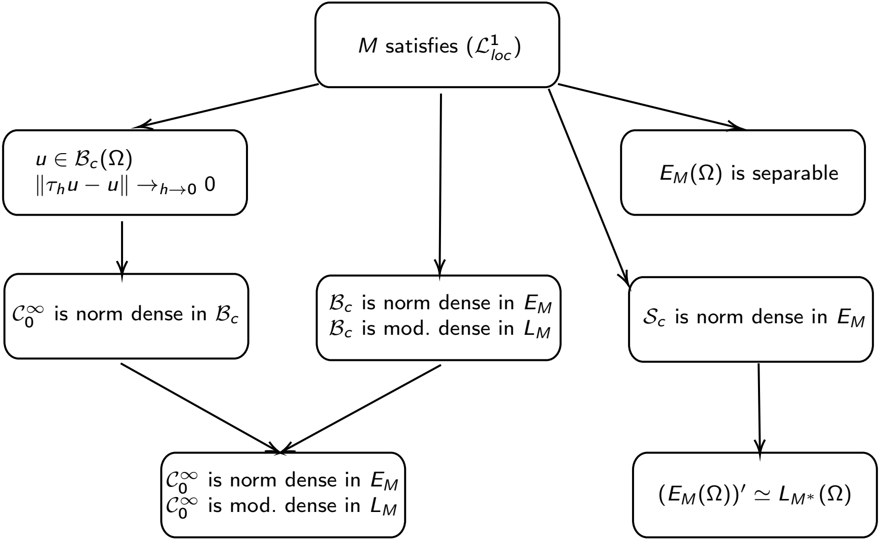

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Part II

Structural assumptions and Poincaré-type inequalities in Musielak-Sobolev spaces

Musielak-Sobolev spaces

For a positive integer m, define

$$W^m L_M(\Omega) = \Big\{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), |\alpha| \leq m\Big\},$$

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Let $M \in \Phi$ satisfies $(\mathcal{L}^1_{loc})$. Define

$$W^1_0 E_M(\Omega) := \overline{\mathcal{C}^\infty_0(\Omega)}^{\|\cdot\|_{W^1 L_M(\Omega)}}$$

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Let $M \in \Phi$ satisfies $(\mathcal{L}^1_{loc})$. Define

$$W^1_0 E_M(\Omega) := \overline{\mathcal{C}^\infty_0(\Omega)}^{\|\cdot\|_{W^1 L_M(\Omega)}}$$

Finally, for a pair (M, M^*) of complementary Φ-functions satisfying booth $(\mathcal{L}^1_{loc})$, we denote

$$W^1_0 L_M(\Omega) := \overline{\mathcal{C}^\infty_0(\Omega)}^{\sigma^*(\Pi L_M, \Pi E_{M^*})}.$$

Lavrentiev phenomenon (1926)

Lavrentiev phenomenon

The Lavrentiev phenomenon occurs when the infimum of the variational problem over the smooth functions is strictly greater than infimum taken over the set of all functions satisfying the same boundary conditions.

Lavrentiev phenomenon (1926)

The situation can be illustrated as follows, Let $H(u) = \int_\Omega h(x, \nabla u)dx$, with integrand $h(x, \xi)$ measurable in $x \in \Omega$ and convex in $\xi \in \mathbb{R}^N$ and satisfies

$$-l_0(x) + k_1|\xi|^{p_1} \leq h(x, \xi) \leq l_0(x) + k_1|\xi|^{p_2}$$

where $l_0 \in L^1(\Omega)$, $k_1 > 0$, $1 < p_1 \leq p_2$. Then the variational problem $H_1 = \inf_{u \in W_0^{1,p_1}} H(u)$ has a solution (Zhikov 2011) Nevertheless, the functional H is not necessarily continuous on $W_0^{1,p_1}(\Omega)$ and it can happen the following inequality

$$\inf_{u \in W_0^{1,p_1}} H(u) < \inf_{u \in \mathcal{C}_0^\infty} H(u).$$

Approximation in the modular sense

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$$u_k \to u \text{ modularly in } W^m L_M(\Omega)$$

when

$$\exists \lambda > 0,\ \forall |\alpha| \leq m, \quad \int_\Omega M\Big(x, \frac{|D^\alpha u_k(x) - D^\alpha u(x)|}{\lambda}\Big)\, dx \xrightarrow[k\to\infty]{} 0.$$

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Do we have

$$W_0^1 L_M(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{mod} ?$$

Enter balanced Φ-functions

There exists a function $\varphi : [0,1/2] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(\cdot, s)$ and $\varphi(x, \cdot)$ are non-decreasing functions and for all $x, y \in \overline{\Omega}$ with $|x-y| \leq \frac{1}{2}$ and for any constant $c > 0$

$$M(x,s) \leq \varphi(|x-y|, s) M(y,s), \quad \text{with } \limsup_{\varepsilon \to 0^+} \varphi(\varepsilon, c\varepsilon^{-N}) < \infty. \qquad (\mathcal{M})$$

Testing on some models

1. Orlicz functions $M(x,s) = M(s)$, not depending on x, satisfy obviously $(\mathcal{M})$ with $\varphi(\tau, s) = 1$.

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$$\varphi(\tau,s) = \max\left\{s^{\sigma(\tau)}, s^{-\sigma(\tau)}\right\}.$$

where $\sigma : (0,1/2] \to \mathbb{R}^+$ is such that $\limsup_{\varepsilon\to 0}\sigma(\varepsilon) = 0$.

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where $\sigma : (0,1/2] \to \mathbb{R}^+$ is such that $\limsup_{\varepsilon\to 0}\sigma(\varepsilon) = 0$. The choice $\sigma(\tau) = -c/\log\tau$, with $0 < \tau \leq 1/2$ yields

$$|p(x) - p(y)| \leq \frac{c}{\log\frac{1}{|x-y|}},\ |x-y| \leq 1/2.$$

Testing on some models

3. $M(x,s)=\frac{1}{p(x)}|s|^{p(x)}$. If $1\le p^-\le p(\cdot)\le p^+<+\infty$. Then M satisfies $(\mathcal{M})$ with
$$\varphi(\tau,s)=\frac{p^+}{p^-}\max\left\{s^{-\frac{c}{\log\tau}},s^{\frac{c}{\log\tau}}\right\}.$$

4. Double phase $M(x,s)=s^p+a(x)s^q$, $1\le p<q$ and $0\le a\in C^{0,\alpha}_{loc}(\Omega)$ with $\alpha\in(0,1]$. Then M satisfies $(\mathcal{M})$ with
$$\varphi(\tau,s)=C_a\tau^{\alpha}|s|^{q-p}+1.$$

$\limsup_{\varepsilon\to 0^+}\varphi(\varepsilon,c\varepsilon^{-N})<\infty$ forces $q\le p+\alpha/N$.

Furthermore, M is called uniformly superlinear if it satisfies

$$\lim_{s \to \infty} ess \inf_{x \in \Omega} \frac{M(x, s)}{s} = +\infty. \qquad (\mathcal{SL})$$

M satisfies $(\mathcal{L}^1_{loc})$

Figure: Some relationships

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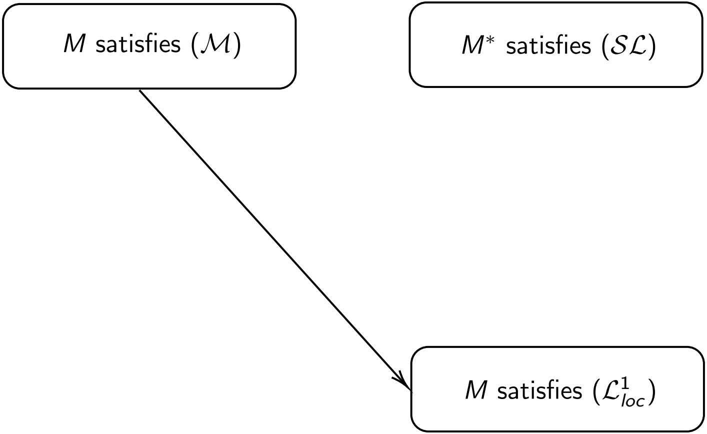

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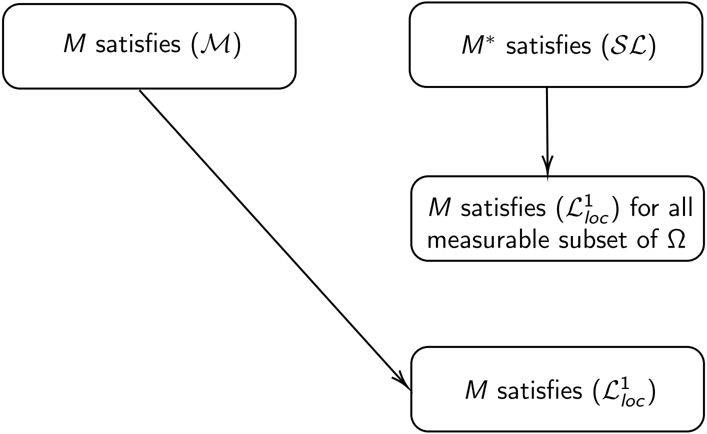

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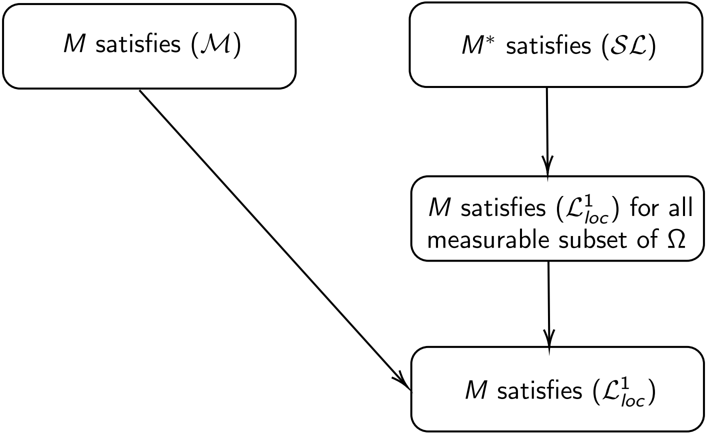

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Poincaré-type inequalities

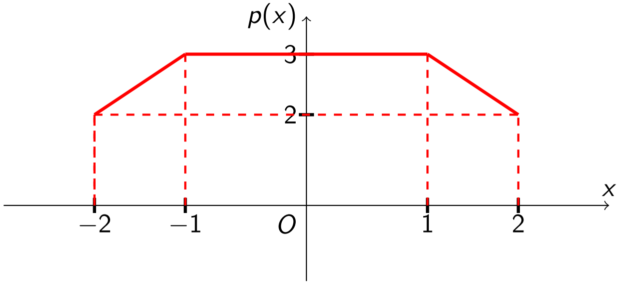
In the paper
X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, JMAA, 263(2)424-446, 2001,
the authors proved that

$$\lambda = \inf_{0 \neq u \in W_{\mathbf{0}}^{\mathbf{1},p(\cdot)}(\Omega)} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx}$$

is equal to zero when $\Omega = (-2, 2) \subset \mathbb{R}$ and

$$p(x) = \begin{cases} 3 & \text{if } 0 \leq |x| \leq 1, \\ 4 - |x| & \text{if } 1 \leq |x| \leq 2. \end{cases}$$

Poincaré-type inequalities



Monotony assumptions

A Φ-function M is said to satisfy the Y-condition on a segment $[a, b]$ of the real line $\mathbb{R}$, if
Either

$$\begin{cases} (\mathcal{M}_{on})_0 : \begin{cases} \text{There exist } t_0 \in \mathbb{R}^+ \text{ and } 1 \leq i \leq N \text{ such that the partial function} \\ x_i \in [a, b] \mapsto M(x, t) \text{ changes constantly its monotony on both} \\ \text{sides of } t_0 \text{ (that is for } t \geq t_0 \text{ and } t < t_0\text{),} \end{cases} \\ \text{Or} \\ (\mathcal{M}_{on})_\infty : \begin{cases} \text{There exists } 1 \leq i \leq N \text{ such that for all } t \geq 0, \text{ the partial} \\ \text{function } x_i \in [a, b] \mapsto M(x, t) \text{ is monotone on } [a, b]. \end{cases} \end{cases} \qquad (\mathcal{M}_{on})$$

x_i is the i^{th} component of $x \in \Omega$.

Some models satisfying $(\mathcal{M}_{on})$

1. $M(x,t) = t^{p(x)}$. The assumption $(\mathcal{M}_{on})_0$ prevents the variable exponent $p(\cdot)$ to get a local extremum while $(\mathcal{M}_{on})_\infty$ is not satisfied unless $p(\cdot)$ is a constant function.

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2. $M(x,t) = t^p + a(x)t^q$. If there is $1 \leq i \leq N$ such that the function $x_i \mapsto a(x)$ is monotone then M satisfies obviously $(\mathcal{M}_{on})_\infty$. If $x_i \mapsto a(x)$ is not a constant function then the double phase function M can not satisfy $(\mathcal{M}_{on})_0$.

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3. If $1 < p(\cdot) < +\infty$ and there exists $1 \leq i \leq N$ such that the function $x_i \mapsto p(x)$ is monotone on a compact subset of the real line $\mathbb{R}$, then the following Φ-functions

$$M_1(x,t) = t^{p(x)}, \qquad M_2(x,t) = t^{p(x)}\log(e+t), \qquad M_3(x,t) = e^{t^{p(x)}} - 1,$$

satisfy $(\mathcal{M}_{on})$.

Poincaré-type inequalities in $W_0^m L_M(\Omega)$

Poincaré-type inequalities in $W_0^m L_M(\Omega)$

Theorem 8

Let Ω be a bounded open subset in $\mathbb{R}^N$ having the segment property. Let M and M^* be a pair of complementary Φ-functions such that M satisfies $(\mathcal{M})$ and $(\mathcal{M}_{on})$ and M^* satisfies $(\mathcal{L}^1_{loc})$. Then there exists a constant $c_{m,\Omega}$ depending only on m and Ω such that for every $u \in W_0^m L_M(\Omega)$

$$\int_\Omega \sum_{|\alpha|<m} M(x, |D^\alpha u|)dx \leq \int_\Omega \sum_{|\alpha|=m} M(x, c_{m,\Omega}|D^\alpha u|)dx. \tag{2}$$

Moreover, for every $u \in W_0^m L_M(\Omega)$

$$\sum_{|\alpha|<m} \|D^\alpha u\|_{M,\Omega} \leq C(m,\Omega) \sum_{|\alpha|=m} \|D^\alpha u\|_{M,\Omega}, \tag{3}$$

where $C(m,\Omega)$ is a constant depending only on m and Ω.

Poincaré-type inequalities in $W_0^m E_M(\Omega)$

Corollary 9

Let Ω be a bounded open subset in $\mathbb{R}^N$ and let $M \in \Phi$ satisfies $(\mathcal{L}^1_{loc})$ and $(\mathcal{M}_{on})$. The inequality (2) holds true for every $u \in W_0^m E_M(\Omega)$ and then so is (3).

Part III

Null trace functions spaces in Musielak-Sobolev spaces

Musielak functions with compact support

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$$K_0^m L_M(\Omega) = \overline{\{u \in W^m L_M(\Omega) : \text{support } u \text{ is compact} \subset \Omega\}}^{\|\cdot\|_{W^m L_M(\Omega)}}$$

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Let $M(x,t) = t^{p(x)}$. Set $W_0^{m,p(\cdot)}(\Omega) := \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{W^{m,p(\cdot)}(\Omega)}} \subset K_0^m L_M(\Omega)$.

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If $\mathcal{C}^\infty(\Omega) \cap W^{m,p(\cdot)}(\Omega)$ is dense in $W^{m,p(\cdot)}(\Omega)$ then $W_0^{m,p(\cdot)}(\Omega) = K_0^m L_M(\Omega)$, (see the book by Diening et al.)

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Theorem 10

Let Ω be an open subset in $\mathbb{R}^N$ and let $M \in \Phi$ satisfies $(\mathcal{M})$. Then $K_0^m L_M(\Omega)$ coincides with $W_0^m E_M(\Omega)$.

Null trace functions space

For every $u \in W^1 L_M(\Omega)$ the trace $tr(u) = u|_{\partial\Omega}$ is well defined.

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$$W^1L_M(\Omega) \underbrace{\hookrightarrow}_{(\mathcal{S}\mathcal{L})+|\Omega|<\infty} W^{1,1}(\Omega) \underbrace{\hookrightarrow}_{Gagliardo\ trace\ thm\ 1957} L^1(\partial\Omega)$$

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For every $u \in W^1L_M(\Omega)$ the trace $tr(u) = u|_{\partial\Omega}$ is well defined.Indeed,

$$W^1L_M(\Omega) \underbrace{\hookrightarrow}_{(\mathcal{SL})+|\Omega|<\infty} W^{1,1}(\Omega) \underbrace{\hookrightarrow}_{\textit{Gagliardo trace thm } 1957} L^1(\partial\Omega)$$

Identify

$$\Big\{u \in W^1L_M(\Omega): \ \mathrm{tr}(u) = 0 \text{ on } \partial\Omega\Big\}.$$

Null trace functions space

Theorem 11

Let Ω be a bounded open subset in $\mathbb{R}^N$ having the segment property and M, M^* be a pair of complementary Φ-functions. Assume that M satisfies $(\mathcal{M})$ and $(\mathcal{SL})$. Then, we get

$$W_0^m L_M(\Omega) = W_0^{m,1}(\Omega) \cap W^m L_M(\Omega).$$

Null trace functions space

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If furthermore Ω has a Lipschitz boundary $\partial\Omega$, then we obtain

$$W_0^1 L_M(\Omega) = \{u \in W^1 L_M(\Omega) : \ tr(u) = 0 \text{ on } \partial\Omega\}. \tag{4}$$

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The End

Thank you for your attention