A Virtual Element Method for a Nonlocal FitzHugh-Nagumo Model of Cardiac Electrophysiology.

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Outline

Model problem

$T > 0$, $\Omega \subset \mathbb{R}^2$ with polygonal boundary Σ. $\Omega_T := \Omega \times (0, T)$; $\Sigma_T := \Sigma \times (0, T)$; $v = v(x,t)$ and $w = w(x,t)$ stand for the transmembrane potential and the gating variable, respectively. The nonlocal reaction-diffusion FitzHugh-Nagumo system is:

$$\begin{cases} \partial_t v - \mathcal{D}\left(\int_\Omega v(x,t)\,dx\right)\Delta v + I_{\mathrm{ion}}(v,w) = I_{\mathrm{app}}(x,t) & (x,t) \in \Omega_T, \\ \partial_t w - H(v,w) = 0 & (x,t) \in \Omega_T, \\ \mathcal{D}\left(\int_\Omega v(x,t)\,dx\right)\nabla v \cdot \boldsymbol{n} = 0 & (x,t) \in \Sigma_T, \\ v(x,0) = v_0(x), \qquad w(x,0) = w_0(x) & x \in \Omega. \end{cases}$$

$I_{\mathrm{app}}(x,t) \in L^2(\Omega_T)$ is the stimulus.

diffusion rate $\mathcal{D}$ and ionic current $I_{\rm ion}$ hipotheses

$\mathcal{D} : \mathcal{R} \to \mathcal{R}$ is a continuous function satisfying

$$d_1 \leq \mathcal{D} \quad \text{and} \quad |\mathcal{D}(l_1) - \mathcal{D}(l_2)| \leq d_2 \, |l_1 - l_2| \quad \text{for all} \quad l_1, l_2 \in \mathcal{R}.$$

$I_{\rm ion}(v, w) = I_{1,\rm ion}(v) + I_{2,\rm ion}(w)$.
$I_{1,\rm ion}, I_{2,\rm ion} : \mathcal{R} \to \mathcal{R}$ and $H : \mathcal{R}^2 \to \mathcal{R}$ are continuous functions,
$\exists \ \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$ such that

$$a) \quad \frac{1}{\alpha_1} |v|^4 \leq |I_{1,\rm ion}(v) v| \leq \alpha_2 \left(|v|^4 + 1 \right),$$

$$b) \quad |I_{2,\rm ion}(w)| \leq \alpha_3 (|w| + 1),$$

$$c) \quad \forall \, z, s \in \mathcal{R} \quad (I_{1,\rm ion}(z) - I_{1,\rm ion}(s))(z - s) \geq -C_h |z - s|^2,$$

$$d) \quad |H(v, w)| \leq \alpha_4 (|v| + |w| + 1).$$

Weak solution

Definition (Weak solution)

A weak solution of the system is a couple (v, w) such that $v \in L^2(0,T;H^1(\Omega)) \cap L^4(\Omega_T)$, $\partial_t v \in L^2(0,T;H^1(\Omega)') + L^{\frac{4}{3}}(\Omega_T)$, $w \in C([0,T];L^2(\Omega))$, satisfying

$\iint_{\Omega_T} \partial_t v\, \varphi + \int_0^T \mathcal{D}\left(\int_\Omega v(x,t)\,dx\right) \int_\Omega \nabla v \cdot \nabla \varphi + \iint_{\Omega_T} I_{\mathrm{ion}}(v,w)\varphi = \iint_{\Omega_T} I_{\mathrm{app}}(x,t)\varphi,$

$\iint_{\Omega_T} \partial_t w\, \phi - \iint_{\Omega_T} H(v,w)\phi = 0,$

for all $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^4(\Omega_T)$ and $\phi \in C([0,T];L^2(\Omega))$.

Remark

Note that, the above Definition implies $v \in C([0,T];L^2(\Omega))$.

The virtual elements

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into polygons K.
h_K is the diameter of the element K and $h := \max_{K \in \mathcal{T}_h} h_K$.

$\exists C_{\mathcal{T}}$ such that, $\forall h$ and $\forall K \in \mathcal{T}_h$,

A1: the ratio between the shortest edge and the diameter h_K of K is larger than $C_{\mathcal{T}}$;

A2: $K \in \mathcal{T}_h$ is star-shaped with respect to every point of a ball of radius $C_{\mathcal{T}} h_K$.

$\forall S \subseteq \mathcal{R}^2$, $k \in \mathbb{N}$, we indicate by $\mathbb{P}_k(S)$ the space of polynomials of degree up to k defined on S.
$\forall K \in \mathcal{T}_h$, the local space $V_{k|K}$ is defined by

$$V_{k|K} := \{\varphi \in H^1(K) \cap C^0(K) : \varphi_{|e} \in \mathbb{P}_k(e) \; \forall e \in \partial K, \; \Delta\varphi \in \mathbb{P}_k(K)\}.$$

Linear operators and bilinear form

B.Ahmad, A.Alsaedi, F.Brezzi, L.D.Marini, A.Russo, Comput. Math. Appl. (2013)

Now, we introduce the following set of linear operators from $V_{k|K}$ into $\mathcal{R}$. For all $\varphi \in V_{k|K}$:

- D_1: The values of φ at the vertices of K;
- D_2: Values of φ at $k-1$ distinct points in e, for all $e \in \partial K$;
- D_3: All moments $\int_K \varphi p \, dx$, for all $p \in \mathbb{P}_{k-2}(K)$.

Now, we split the bilinear form $a(\cdot,\cdot) := (\nabla \cdot, \nabla \cdot)_{0,\Omega}$,

$$a(v,\varphi) := \sum_{K \in \mathcal{T}_h} a^K(v,\varphi), \qquad \forall v,\varphi \in H^1(\Omega),$$

where

$$a^K(v,\varphi) := \int_K \nabla v \cdot \nabla \varphi, \qquad \forall v,\varphi \in H^1(\Omega).$$

The projection operator

Let $\Pi_{K,k} : V_{k|K} \to \mathbb{P}_k(K)$ be the projection operator defined by

$$\begin{cases} a^K(\Pi_{K,k} v, q) = a^K(v, q) & \forall q \in \mathbb{P}_k(K), \\ P_0(\Pi_{K,k} v) = P_0 v, \end{cases}$$

where P_0 can be taken as

$$\begin{cases} P_0 v := \frac{1}{N_K} \sum_{i=1}^{N_K} v(V_i) & k = 1, \\ P_0 v := \frac{1}{|K|} \int_K v \, dx & k > 1, \end{cases}$$

V_i the vertices of K, $1 \leq i \leq N_K$ (N_K number of vertices).
For any $\varphi \in V_{k|K}$, the values of the linear operators D_1, D_2 and D_3 are sufficient to compute $\Pi_{K,k}$.
As a consequence, the projection operator $\Pi_{K,k}$ depends only on the values of the operators D_1, D_2 and D_3.

The virtual local space

B.Ahmad, A.Alsaedi, F.Brezzi, L.D.Marini, A.Russo, Comput. Math. Appl. (2013)

Now, we introduce our virtual local space

$$W_{k|K} := \left\{ \varphi \in V_{k|K} : \int_K (\Pi_{K,k}\varphi) q \, dx = \int_K \varphi q \, dx \quad \forall q \in \mathbb{P}_k/\mathbb{P}_{k-2}(K) \right\},$$

$\mathbb{P}_k/\mathbb{P}_{k-2}(K)$ polynomials of degree k, L^2-orthogonal to polynom. of degree $k-2$ on K.
Since $W_{k|K} \subset V_{k|K}$, the operator $\Pi_{K,k}$ is well defined on $W_{k|K}$ and computable on the basis of the values of the operators D_1, D_2, D_3.
The operators D_1, D_2 and D_3 constitute a set of degrees of freedom for the space $W_{k|K}$.

Global degrees of freedom

The global discrete space will be

$$W_h := \{\varphi \in H^1(\Omega) : \varphi|_K \in W_{k|K}, \quad \forall K \in \mathcal{T}_h\}.$$

In agreement with the local choice of the degrees of freedom, in W_h we choose the following degrees of freedom:

- DG_1: The values of φ at the vertices of $\mathcal{T}_h$;
- DG_2: Values of φ at $k-1$ distinct points in e, for all $e \in \mathcal{T}_h$;
- DG_3: All moments $\displaystyle\int_K \varphi p \, dx$, for all $p \in \mathbb{P}_{k-2}(K)$ on each element $K \in \mathcal{T}_h$.

Ancillary symmetric positive definite bilinear forms

On the other hand, let $S^K(\cdot,\cdot)$ and $S_0^K(\cdot,\cdot)$ be any symmetric positive definite bilinear forms to be chosen as to satisfy

$$c_0 a^K(\varphi_h,\varphi_h) \leq S^K(\varphi_h,\varphi_h) \leq c_1 a^K(\varphi_h,\varphi_h) \forall \varphi_h \in V_{k|K} \quad with \quad \Pi_{K,k}\varphi_h = 0,$$

$$\tilde{c}_0(\varphi_h,\varphi_h)_{0,K} \leq S_0^K(\varphi_h,\varphi_h) \leq \tilde{c}_1(\varphi_h,\varphi_h)_{0,K} \forall \varphi_h \in V_{k|K},$$

for some positive constants $c_0, c_1, \tilde{c}_0$ and $\tilde{c}_1$ independent of K.

Discrete bilinear and trilinear forms

We define the local discrete bilinear and trilinear forms:

$$a_h^K(\cdot,\cdot) : W_h \times W_h \to \mathcal{R}, \qquad m_h^K(\cdot,\cdot) : W_h \times W_h \to \mathcal{R},$$

$$b_h^K(\cdot,\cdot,\cdot) : W_h \times W_h \times W_h \to \mathcal{R}, \quad c_h^K(\cdot,\cdot,\cdot) : W_h \times W_h \times W_h \to \mathcal{R},$$

Discrete bilinear and trilinear forms

As follow, for all $v_h, w_h, \varphi_h \in W_{k|K}$:

$$a_h^K(v_h, \varphi_h) := a^K(\Pi_{K,k} v_h, \Pi_{K,k}\varphi_h) + S^K(v_h - \Pi_{K,k} v_h, \varphi_h - \Pi_{K,k}\varphi_h),$$
$$m_h^K(v_h, \varphi_h) := (\Pi^0_{K,k} v_h, \Pi^0_{K,k}\varphi_h) + S_0^K(v_h - \Pi^0_{K,k} v_h, \varphi_h - \Pi^0_{K,k}\varphi_h),$$
$$b_h^K(v_h, w_h, \varphi_h) := \int_K I_{ion}(\Pi^0_{K,k} v_h, \Pi^0_{K,k} w_h)\Pi^0_{K,k}\varphi_h,$$
$$c_h^K(v_h, w_h, \varphi_h) := \int_K H(\Pi^0_{K,k} v_h, \Pi^0_{K,k} w_h)\Pi^0_{K,k}\varphi_h,$$

where $\Pi^0_{K,k} : W_{k|K} \to \mathbb{P}_k(K)$ is the standard L^2-projection operator. We note that all the forms introduced above are computable on the basis of the degrees of freedom.

Consistency and stability

We observe that for all $K \in \mathcal{T}_h$ it holds:

- k-consistency: for all $p \in \mathbb{P}_k(K)$ and for all $\varphi_h \in W_{k|K}$

$$a_h^K(p, \varphi_h) = a^K(p, \varphi_h),$$
$$m_h^K(p, \varphi_h) = (p, \varphi_h)_{0,K}.$$

- stability: there exist four positive constants $\alpha', \alpha'', \beta', \beta''$, independent of h, such that for all $\varphi_h \in W_{k|K}$

$$\alpha' \; a^K(\varphi_h, \varphi_h) \leq a_h^K(\varphi_h, \varphi_h) \leq \alpha'' \; a^K(\varphi_h, \varphi_h),$$
$$\beta' \; (\varphi_h, \varphi_h)_{0,K} \leq m_h^K(\varphi_h, \varphi_h) \leq \beta'' \; (\varphi_h, \varphi_h)_{0,K}.$$

Then, we set for all $v_h, w_h, \varphi_h \in W_h$,

$$a_h(v_h, \varphi_h) := \sum_{K \in \mathcal{T}_h} a_h^K(v_h, \varphi_h), \quad m_h(v_h, \varphi_h) := \sum_{K \in \mathcal{T}_h} m_h^K(v_h, \varphi_h),$$

Nonlocal diffusion and nonlinear right-hand side

We discretize the nonlocal diffusion term using the L^2-projection

$$J(v_h) := \int_\Omega v_h = \sum_{K \in \mathcal{T}_h} \int_K \Pi^0_{K,k} v_h, \qquad v_h \in W_h.$$

For the right-hand side, since $I_{\mathrm{app}}(x,t) \in L^2(\Omega_T)$, we set

$$I_{app,h}(t) = \Pi^0_k I_{\mathrm{app}}(\cdot, t) \quad for\ a.e. \quad t \in (0,T),$$

where Π^0_k is defined by

$$(\Pi^0_k g)|_K := \Pi^0_{K,k} g \qquad \text{for all } K \in \mathcal{T}_h$$

with $\Pi^0_{K,k}$ the $L^2(K)$-projection.

Continuity

Now, we note that the symmetry of $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, and the stability conditions stated before imply the continuity of a_h and m_h. In fact, for all $v_h, \varphi_h \in W_h$:

$$|a_h(v_h,\varphi_h)| \leq C\|v_h\|_{H^1(\Omega)}\|\varphi_h\|_{H^1(\Omega)},$$
$$|m_h(v_h,\varphi_h)| \leq C\|v_h\|_{L^2(\Omega)}\|\varphi_h\|_{L^2(\Omega)}.$$

The semidiscrete scheme

The semidiscrete VEM formulation reads as follows. For all $t > 0$, find $v_h, w_h \in L^2(0, T; W_h)$ with $\partial_t v_h, \partial_t w_h \in L^2(0, T; W_h)$, such that

$$\begin{cases} m_h(\partial_t v_h(t), \varphi_h) & + \quad \mathcal{D}\left(J(v_h(t))\right) a_h\Big(v_h(t), \varphi_h\Big) \\ & + \quad b_h(v_h(t), w_h(t), \varphi_h) \;=\; (I_{app,h}(t), \varphi_h)_{0,\Omega} \\ m_h(\partial_t w_h(t), \phi_h) & - \quad c_h(v_h(t), w_h(t), \phi_h) \;=\; 0, \end{cases}$$

for all $\varphi_h, \phi_h \in W_h$. Additionally, we set $v_h(0) = v_h^0$ and $w_h(0) = w_h^0$. A classical backward Euler integration method is employed for the time discretization with time step $\Delta t = T/N$.

The fully discrete scheme

We set $v_h(0) = v_h^0$ and $w_h(0) = w_h^0$. Find $v_h^n, w_h^n \in W_h$ such that

$$\left\{ \begin{aligned} & m_h\left(\frac{v_h^n - v_h^{n-1}}{\Delta t}, \varphi_h\right) + \mathcal{D}\left(J(v_h^n)\right) a_h\Big(v_h^n, \varphi_h\Big) + b_h(v_h^n, w_h^n, \varphi_h) \\ & \qquad = \Big(I_{app,h}^n, \varphi_h\Big)_{0,\Omega} \\ & m_h\left(\frac{w_h^n - w_h^{n-1}}{\Delta t}, \phi_h\right) - c_h(v_h^n, w_h^n, \phi_h) = 0, \end{aligned} \right.$$

for all $\varphi_h, \phi_h \in W_h$, for all $n \in \{1, \dots, N\}$; $I_{app,h}^n := I_{app,h}(t_n)$ with $t_n := n\Delta t$, for $n = 0, \dots, N$. We denote

$$v_h := \sum_{n=1}^{N} v_h^n(x) \mathbb{1}_{((n-1)\Delta t, n\Delta t]}(t), \quad w_h := \sum_{n=1}^{N} w_h^n(x) \mathbb{1}_{((n-1)\Delta t, n\Delta t]}(t).$$

Existence of solution for the virtual element scheme

Proposition

Under the previous assumptions, the full numerical scheme admits a discrete solution $\mathbf{u}_h^n = (v_h^n, w_h^n)$.

Proof: The existence of $\mathbf{u}_h^n$ is shown by induction on $n = 0, \ldots, N$. For $n = 0$, solution is given by $\mathbf{u}_h^0 = (v_h(0), w_h(0)) = (v_h^0, w_h^0)$. Assume that $\mathbf{u}_h^{n-1}$ exists. Choose $[\![\cdot\,,\,\cdot]\!]$ as the scalar product on $H^1(\Omega) \times L^2(\Omega)$.

Proof of existence of solution for the virtual element scheme

We define a map $L : W_h \times W_h \to W_h \times W_h$ such that for every $\mathbf{u}_h^n \in W_h \times W_h$, $L(\mathbf{u}_h^n) \in W_h \times W_h$ is the solution of following problem:

$$\begin{aligned}
[\![L(\mathbf{u}_h^n), \Phi_h]\!] = & m_h\left(\frac{v_h^n - v_h^{n-1}}{\Delta t}, \varphi_h\right) + \mathcal{D}\left(J(v_h^n)\right) a_h(v_h^n, \varphi_h) \\
& + b_h(v_h^n, w_h^n, \varphi_n) - (I_{app,h}(t_n), \varphi_h)_{0,\Omega} \\
& + m_h\left(\frac{w_h^n - w_h^{n-1}}{\Delta t}, \phi_h\right) - c_h(v_h^n, w_h^n, \phi_h),
\end{aligned}$$

for all $\Phi_h := (\varphi_h, \phi_h) \in W_h \times W_h$.

Proof of existence of solution for the virtual element scheme

Next, we are looking for a solution $\mathbf{u}_h^n$ to $\llbracket L(\mathbf{u}_h^n), \Phi_h \rrbracket = 0$. Note that the continuity of the operator L is a consequence of the continuity of m_h, a_h b_h and c_h. Moreover, the following bound holds from the discrete Hölder and Sobolev inequalities (recall that $H^1(\Omega) \subset L^q(\Omega)$ for all $1 \leq q \leq 6$):

$$\llbracket L\big(\mathbf{u}_h^n\big), \Phi_h \rrbracket \leq C(\|v_h^n\|_{H^1(\Omega)} + \|w_h^n\|_{L^2(\Omega)} + 1)(\|\varphi_h\|_{H^1(\Omega)} + \|\phi_h\|_{L^2(\Omega)}),$$

for all $\mathbf{u}_h^n$ and Φ_h in $W_h \times W_h$.

Proof of existence of solution for the virtual element scheme

Using Young inequality, we get

$$[\![L(\mathbf{u}_h^n), \mathbf{u}_h^n]\!] \geq C(\|v_h^n\|_{H^1(\Omega)}^2 + \|w_h^n\|_{L^2(\Omega)}^2) + C'$$

for some constants $C > 0$ and C'. Finally, we conclude that $[\![L(\mathbf{u}_h^n)\,,\, \mathbf{u}_h^n]\!] \geq 0$ for $\|\mathbf{u}_h^n\|^2 := \|v_h^n\|_{H^1(\Omega)}^2 + \|w_h^n\|_{L^2(\Omega)}^2$ sufficiently large. The existence of $\mathbf{u}_h^n$ follows by the standard Brouwer fixed point argument. $\square$

A priori estimates

Proposition

Let $\mathbf{u}_h^n = (v_h^n, w_h^n)$ be a solution of the virtual element scheme (18). Then, there exist constants $C > 0$, depending on Ω, T, v_h^0, w_h^0, I_{app} and α_i, with $i = 1, \dots 4$, such that

$$\|v_h\|_{L^\infty(0,T;L^2(\Omega))} + \|w_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C,$$
$$\|\nabla v_h\|_{L^2(\Omega_T)} \leq C, \qquad \|\Pi_k^0 v_h\|_{L^4(\Omega_T)} \leq C,$$

Idea of the proof: Energy method. Cauchy-Schwarz, Young, and discrete Gronwall inequalities.

$$\frac{1}{2}\beta'\|v_h^\kappa\|^2_{L^2(\Omega)} + \frac{1}{2}\beta'\|w_h^\kappa\|^2_{L^2(\Omega)} + d_1\alpha'\int_0^{\kappa\Delta t}|v_h|^2_{H^1(\Omega)} + \sum_{n=1}^{\kappa}\Delta t\left(\sum_{K\in\mathcal{T}_h}\int_K\frac{1}{\alpha_1}|\Pi^0_{K,k}v_h^n|^4\right)$$
$$\leq \frac{1}{2}\beta''\|v_h^0\|^2_{L^2(\Omega)} + \frac{1}{2}\beta''\|w_h^0\|^2_{L^2(\Omega)} + C_1\|v_h\|^2_{L^2(\Omega_T)} + C_2\|w_h\|^2_{L^2(\Omega_T)} + C,$$

Interpolation in $W^{1,\infty}([0,T];W_h)$

We introduce $\bar{v}_h$ and $\bar{w}_h$ the piecewise affine in t functions in $W^{1,\infty}([0,T];W_h)$ interpolating the states $(v_h^n)_{n=0,\ldots,N} \subset W_h$ and $(w_h^n)_{n=0,\ldots,N} \subset W_h$ at the points $(n\Delta t)_{n=0,\ldots,N}$. Then, we have

$$\left\{ \begin{array}{l} m_h(\partial_t \bar{v}_h(t),\varphi_h) + \mathcal{D}\left(J(v_h(t))\right) a_h(v_h(t),\varphi_h) \\ +b_h(v_h(t),w_h(t),\varphi_h) = (I_{app,h}(t),\varphi_h)_{0,\Omega}\,, \\ m_h(\partial_t \bar{w}_h(t),\phi_h) = c_h(v_h(t),w_h(t),\phi_h), \end{array} \right.$$

for all φ_h and $\phi_h \in W_h$.

Space Time translate

Lemma

There exists a positive constant $C > 0$ depending on Ω, T, v_0 and I_{app} such that

$$\iint_{\Omega_{\boldsymbol{r}}\times(0,T)} m_h\Big(v_h(x+\boldsymbol{r},t)-v_h(x,t), v_h(x+\boldsymbol{r},t)-v_h(x,t)\Big)\leq C\,|\boldsymbol{r}|^2,$$

for all $\boldsymbol{r} \in \mathbb{R}^2$ with $\Omega_{\boldsymbol{r}} := \{x \in \Omega \,|\, x + \boldsymbol{r} \in \Omega\}$, and

$$\iint_{\Omega\times(0,T-\tau)} m_h\Big(v_h(x,t+\tau)-v_h(x,t), v_h(x,t+\tau)-v_h(x,t)\Big)\,dx\,dt \leq C(\tau+\Delta t), \qquad \text{for all } \tau \in (0,T).$$

Kolmogorov's compactness criterion

Lemma

There exists a subsequence of $\mathbf{u}_h = (v_h, w_h)$, not relabeled, such that, as $h \to 0$,

$$
\begin{aligned}
&v_h, \Pi_k^0 v_h \to v \text{ strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T,\\
&w_h, \Pi_k^0 w_h \to w \text{ weakly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T,\\
&v_h \rightharpoonup v \text{ weakly in } L^2(0,T;H^1(\Omega)),\\
&\Pi_k^0 v_h \rightharpoonup v \text{ weakly in } L^4(\Omega_T).
\end{aligned}
$$

Convergence result

Theorem

Under the previous assumptions, if $v_0(x) \in L^2(\Omega)$, $w_0(x) \in L^2(\Omega)$, and $I_{\mathrm{app}}(x,t) \in L^2(\Omega_T)$, then the virtual element solution $\mathbf{u}_h^n = (v_h^n, w_h^n)$, generated by discrete numercal scheme, converges along a subsequence to $\mathbf{u} = (v, w)$ as $h \to 0$, where $\mathbf{u}$ is a weak solution of the system of equations de the FitzHugh-Nagumo model. Moreover, the weak solution is unique.

Projection $\mathcal{P}^h : H^1(\Omega) \to W_h$

We assume that $I_{\rm ion}$ is a linear function on v and w satisfying

$$\forall\, s_1, s_2, z_1, z_2 \in \mathcal{R} \quad |I_{\rm ion}(s_1, z_1) - I_{\rm ion}(s_2, z_2)| \leq \alpha_7(|s_1 - s_2| + |z_1 - z_2|),$$

for some constant $\alpha_7 > 0$.
First, we introduce the projection $\mathcal{P}^h : H^1(\Omega) \to W_h$ as the solution of the following well-posed problem:

$$\begin{cases} \mathcal{P}^h u \in W_h, \\ a_h(\mathcal{P}^h u, \varphi_h) = a(u, \varphi_h) \text{ for all } \varphi_h \in W_h. \end{cases}$$

Error estimate of the Projection $\mathcal{P}^h : H^1(\Omega) \rightarrow W_h$

L.Beirão da Veiga, F.Brezzi, L.D.Marini and A.Russo, M3AS (2016)

Lemma

Let $u \in H^1(\Omega)$. Then, there exist $C, \tilde{C} > 0$, independent of h, such that

$$\left| \mathcal{P}^h u - u \right|_{H^1(\Omega)} \leq C h^k \left| u \right|_{H^{k+1}(\Omega)} ,$$

Moreover, if the domain is convex, then

$$\left\| \mathcal{P}^h u - u \right\|_{L^2(\Omega)} \leq \tilde{C} h^{k+1} \left| u \right|_{H^{k+1}(\Omega)} .$$

Error estimate result

Theorem

Let (v, w) be the solution of system and let let $\mathbf{u}_h^n = (v_h^n, w_h^n)$ be the virtual element solution generated by the full discrete numerical scheme. Then, for $n = 1, \dots, N$

$$\begin{aligned}
&\left\| v_h^n - v(\cdot, t_n) \right\|_{L^2(\Omega)} + \left\| w_h^n - w(\cdot, t_n) \right\|_{L^2(\Omega)} \\
&\quad \leq C \Bigg[\left\| v_0 - v_h^0 \right\|_{L^2(\Omega)} + \left\| w_0 - w_h^0 \right\|_{L^2(\Omega)} + \Delta t \int_0^{t_n} \left(\left| \partial_{tt}^2 v \right| + \left| \partial_{tt}^2 w \right| \right) dt \\
&\qquad + h^{k+1} \Bigg(|v_0|_{H^{k+1}(\Omega)} + |w_0|_{H^{k+1}(\Omega)} \\
&\quad + \int_0^{t_n} \left(|I_{\mathrm{app}}|_{H^{k+1}(\Omega)} + |v|_{H^{k+1}(\Omega)} + |w|_{H^{k+1}(\Omega)} + |\partial_t v|_{H^{k+1}(\Omega)} + |\partial_t w|_{H^{k+1}(\Omega)} \right) dt \Bigg) \Bigg] \\
&\qquad \times \exp\left(\int_0^{t_n} \left(1 + |v|_{H^2(\Omega)} \right) dt \right)
\end{aligned}$$

Sample meshes

We choose

$$\begin{aligned} H(v,w) &= av - bw, \\ I_{\mathrm{ion}}(v,w) &= -\lambda(w - v(1-v)(v-\theta)), \end{aligned}$$

For each polygon K with vertices $P_1, \dots, P_{N_K}$, we have used

$$S^K(u,v) := \sum_{r=1}^{N_K} u(P_r)v(P_r), \qquad u,v \in W_{1|K},$$

$$S_0^K(u,v) := h_K^2 \sum_{r=1}^{N_K} u(P_r)v(P_r), \qquad u,v \in W_{1|K}.$$

Sample meshes

We test the method by using different families of meshes.

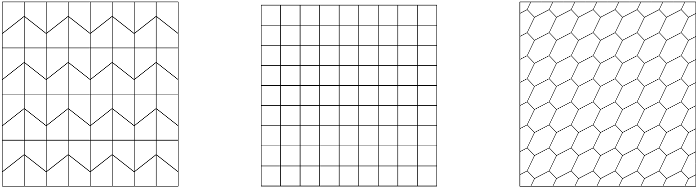


Figure: Sample meshes: $\mathcal{T}_h^1$ (left), $\mathcal{T}_h^2$ (center), $\mathcal{T}_h^3$ (right).

Test 1

$\Omega = (0,1)^2$ and the time interval $[0,1]$,
$a = 0.2232, b = 0.9, \lambda = -1, \theta = 0.004$.
$I_{\mathrm{app}} = 0$ and $D(x) = 0.01x$.
Initial data:

$$v_0(x,y) = (1 + 0.5\cos(4\pi x)\cos(4\pi y)),$$
$$w_0(x,y) = (1 + 0.5\cos(8\pi x)\cos(8\pi y)).$$

We compute errors using a numerical solution on an extremely fine mesh ($h = 1/512$) and time step ($\Delta t = 1/512$) as reference.

Tables of error for v and w

$h\backslash\Delta t$	$\Delta t = 1/3$	$\Delta t = 1/12$	$\Delta t = 1/48$	$\Delta t = 1/192$
1/8	0.523499772859947	0.254128190031018	0.231625702484074	0.228564582239788
1/16	0.501427757954840	0.073397686413675	0.033438153244729	0.031719551242699
1/32	0.499619638795241	0.063643322905268	0.010299560779982	0.005840961963621
1/64	0.499780908876156	0.064056553619930	0.009767337053892	0.002546001572083

Table: Test 1: $E_{h,\Delta t}$ error for v and for the meshes $\mathcal{T}_h^2$.

$h\backslash\Delta t$	$\Delta t = 1/3$	$\Delta t = 1/12$	$\Delta t = 1/48$	$\Delta t = 1/192$
1/8	0.233922447286499	0.102194576523503	0.086875203270260	0.084789535586885
1/16	0.226571951589132	0.089790111454289	0.075921461474953	0.074408607847462
1/32	0.210582296617939	0.049672099006078	0.023584319200822	0.020543068189885
1/64	0.207657184653963	0.043302505350623	0.011225579353452	0.005588513008340

Table: Test 1: $E_{h,\Delta t}$ error for w and for the meshes $\mathcal{T}_h^2$.

Tables of error for v and w

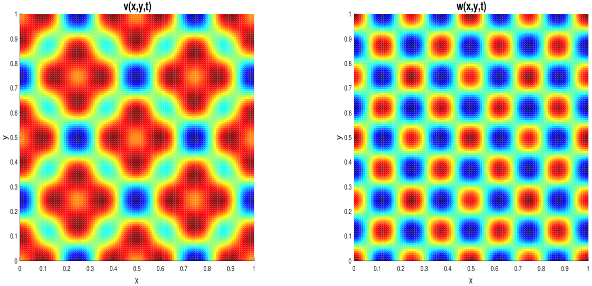


Figure: Test 1: Variables v (left) and w (right) for $h = 1/64$ and $\Delta t = 1/80$.

Test 2: Periodic spiral wave

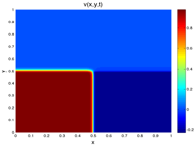
F. Liu, P. Zhuang, I. Turner, V. Anh and K. Burrage, J. Comput. Phys. (2015)

We use meshes $\mathcal{T}_h^3$ (with $h = 1/128$).
$\Omega := (0,1)^2$, and time interval $[0,15]$ (with $\Delta t = 1/200$).
$a = 0.16875, b = 1, \lambda = -100, \theta = 0.25.$
Initial data:

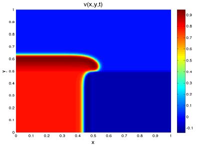
$$v_0(x,y) = \begin{cases} 1.4 & \textit{if} \quad x < 0.5 \quad \textit{and} \quad y < 0.5 \\ 0 & \textit{otherwise}, \end{cases}$$

$$w_0(x,y) = \begin{cases} 0.15 & \textit{if} \quad x > 0.5 \quad \textit{and} \quad y < 0.5 \\ 0 & \textit{otherwise}. \end{cases}$$

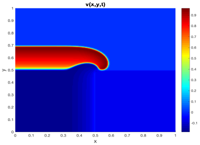
Test 2: Periodic spiral wave



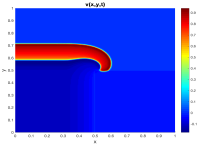
$t = 0.1$



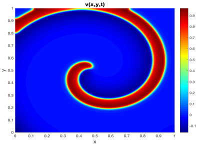
$t = 1.0$



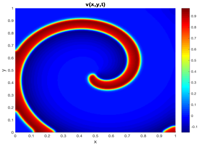
$t = 1.5$



$t = 2.0$



$t = 10.0$



$t = 15.0$

Figure: Test 2: Numerical solution of the transmembrane potential v for different times.

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