On the Eikonal Equation

and
Some Related Problems

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Hamilton-Jacobi equation

We consider

$$H(x, \nabla u) = 0, \quad \text{in } \Omega$$

where

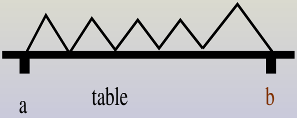
- $\Omega \subset \mathbb{R}^N$ be a bounded domain
- $H : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function (the Hamiltonian) satisfying :

(H1) $Z(x) := \left\{ p \in \mathbb{R}^N \,;\, H(x,p) \leq 0 \right\}$ is **convex,** for any $x \in \Omega$
(H2) $Z(x)$ is compact, for any $x \in \Omega$
(H3) $H(x,0) \leq 0$, for any $x \in \Omega$.

Main example : Eikonal Equation

$$\|\nabla u\| = 1 \tag{1}$$

- For the Eikonal equation, we can construct infinity $W^{1,\infty}$ function null on the boundary of Ω and satisfying $\|\nabla u\| = 1$ a.e. in Ω :



- The right notion of solution of the Eikonal equation is important in order to describe the physical problem.
- The right concept of solution needs to handle :
  - $1-$**Lipschitz continuity**
  - the "maximality" ? ? ? ?

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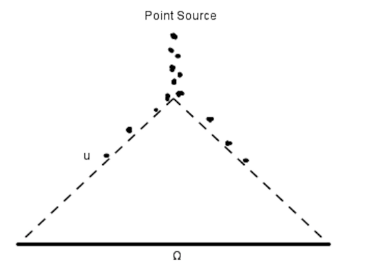
Concept of viscosity solution

Eikonal equation and Sandpile

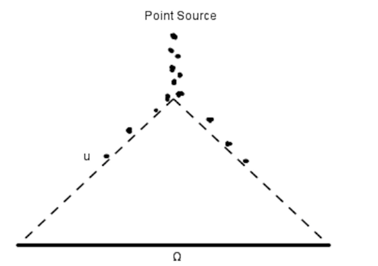
Reminder on Sandpile



- A sandpile is a generic term for any structure of granular materials : *a collection of macroscopic grains large enough such that the Brownien motion is non-existent.*
- The more common property : the ability to get into a slope effect up to the so called repose angle : 45^o *for a pile of gravel or wet sand,* 30^o *for the dry sand,* 38^o *for a pile of snow,* 22^o *for a pile of glass beads,* 15^o *for wet clay and may tends towards* 0^o *to represent fluid material like water.*

- Simplest situation : an homogeneous granular matter is poured continuously onto a flat horizontal table which stands for a flat ground. In this situation, one gets circular cone whereby the slop is determined by the angle of repose of the material considered. The cone grows until its foot reach the boundary of the table and/or any region from where the material can pour out, like holes, reft etc. Then all additional sand runs over the edge in touch with the boundary and goes outside the table. We call this final overflowing geometrical figure the equilibrium.
- The repose angle will be given by its tangent k
- A sandpile can be seen as a surface representation in $\mathbb{R}^3$ of a k-Lipschitz continuous function.

Eikonal equation and Sandpile



The equilibrium corresponds to k-Lipschitz continuous function with the maximal volume (assume that $k = 1$).

- Let $\Omega \subseteq \mathbb{R}^2$ a bounded open domain to represent the table
- A closed $C \subset \overline{\Omega}$ to represent a region from where the sand can run out (for instance $C = \partial\Omega$ or $C = \{y\}$ for a given $y \in \overline{\Omega}$).
- The equilibrium can be seen as a solution of the following maximization volume problem :

$$\max \left\{ \int_\Omega z\, dx \; ; \; z \in K \right\}, \tag{2}$$

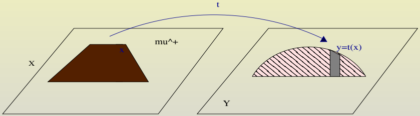
where

$$K := \Big\{ z \in Lip(\Omega) \; ; \; z = 0 \text{ on } C \text{ and } \|\nabla z\| \leq k \text{ a.e. in } \Omega \Big\}.$$

Eikonal Equation and Optimal transportation

Reminder : Monge and Monge-Kantorovich problems

Monge problem :

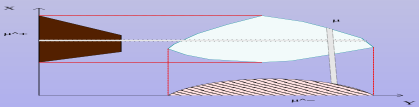


$$\mu^+(\Omega) = \mu^-(\Omega)$$

- $\mathcal{A}(\mu^+,\mu^-) = \left\{ t \ : \ X \to Y \ ; \ \mu^+_{\#t} = \mu^-, \text{ i.e.} \mu^-(B) = \mu^+(t^{-1}(B)) \right\}.$
- Find a map $T^* \in \mathcal{A}(f_1, f_2)$ (called an *optimal map*)

$$(M) \qquad \mathcal{F}_c(T^*) = \min_{t \in \mathcal{A}(f_1,f_2)} \mathcal{F}_c(t) \quad \text{where} \quad \mathcal{F}_c(t) = \int_X c(x,t(x)) f_1(x)\, dx$$

Monge-Kantorovich problem :



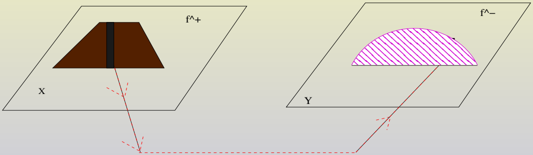
$$\mu^+ = proj_x \mu, \quad \mu^- = proj_y \mu$$

- $\pi(\mu^+,\mu^- 2) := \left\{ \mu \in \mathcal{M}_b(X \times Y) \ ; \ \mu^+ = proj_x \mu \text{ and } \mu^- = proj_y \mu \right\}$
- Find a measure $\mu^* \in \pi(f_1, f_2)$ (called an *optimal transport plan*)

$$(MK) \qquad K_c(\mu^*) := \min_{\mu \in \pi(f\mu^+,\mu^-)} K_c(\mu), \text{ where } K_c(\mu) = \int c(x,y)\, d\mu(x,y)$$

Reminder : Kantorovich dual problem

<u>**Kantorovich dual problem**</u> :



$\mu^+(x)u(x)$ ... $\mu^-(x)v(y)$

$$\mathcal{D}(u,v) := \int u\,d\mu^+ + \int v\,d\mu^-$$

- Find (u^*, v^*) that maximize

  (DMK) $$\max_{(u,v)\in\Phi_c(\mu^+,\mu^-)} \mathcal{D}(u,v)$$

  where

$$\Phi_c(\mu^+,\mu^-) := \Big\{(u,v) \in L^1_{\mu^+}(\mathbb{R}^N) \times L^1_{\mu^-}(\mathbb{R}^N) \,:\, u(x)+v(y) \le c(x,y)$$

$$\mu^+\text{-a.e. } x \text{ and } \mu^-\text{-a.e. } y\Big\}.$$

Duality for optimal Transportation

Proposition (cf. [Villani]) *Let c be a l.s.c. cost function and $\mu^{\pm} \in \mathcal{M}_b^+(\mathbb{R}^N)$ be two non-negative Radon measures satisfying $\mu^+(\mathbb{R}^N) = \mu^-(\mathbb{R}^N)$. We have*

1. *The Monge-Kantorovich problem has at least one optimal plan and the Kantorovich duality holds to be true, i.e.*

$$\begin{aligned} &\min \{\mathcal{K}(\gamma) \ : \gamma \in \pi(\mu,\nu)\} \\ &= \sup\left\{\mathcal{D}(u,v) := \int_{\mathbb{R}^N} u\, d\mu + \int_{\mathbb{R}^N} v\, d\nu \ : \ (u,v) \in \Phi_c(\mu,\nu)\right\}. \end{aligned} \tag{3}$$

2. *It does not change the value of the supremum in the right-hand side of (3) if one restricts the definition of $\mathcal{S}_c(\mu,\nu)$ to those functions (u,v) which are bounded and continuous.*

3. *If the cost function satisfies the* **triangle inequality,** *the Kantorovich dual problem can be rewritten as*

$$\sup\left\{\int_{\mathbb{R}^N} u\, d(\mu^- - \mu^+) \ : \ u \in Lip_c\right\},$$

   where $Lip_c := \left\{u \ : \ \mathbb{R}^N \mapsto \mathbb{R} \ : \ u(y) - u(x) \le c(x,y)\right\}$. A solution of the Kantorovich dual problem is called Kantorovich potential.

Eikonal equation and Optimal Transportation

Modified Optimal mass transport problem

- $\pi(\mu^+) := \Big\{ \mu \in \mathcal{M}_b(\Omega \times C) \ ; \ \mu^+ = proj_x \mu \Big\}$
- Find a measure $\mu^* \in \pi(\mu^+)$

$$(MK_d) \qquad K_c(\mu^*) := \min_{\mu \in \pi(\mu^+)} K_c(\mu) := \int c(x,y)\, d\mu(x,y)$$

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Optimal transportation of all the mass towards the region C

Assume that

- $c(x,y) = \|x-y\|$, for any $x, y \in \mathbb{R}^N$.
- $\mu^+ = \chi_\Omega$

Theorem *(cf. [Ig&Mazon&al,2014], [Santambrogio&al,2018], [Ig&al,2019] The Kantorovich potential for the optimal transportation of the mass μ^+ into the region C is given by*

$$\max \left\{ \int_\Omega z\, dx \ ; \ z \in K \right\}, \tag{4}$$

here

$$K := \Big\{ z \in Lip(\Omega) \ ; \ z = 0 \text{ on } C \text{ and } \|\nabla z\| \le 1 \text{ a.e. in } \Omega \Big\}.$$

Suitable concept of solution : maximization volume problem

DEFINITION *Let $u \in \mathcal{C}(\overline{\Omega})$ be locally Lipschitz-continuous and let $C \subset \overline{\Omega}$ a bounded closed subset.*

- *u is said to be an a.e. subsolution of $\|\nabla u\| = 1$, in Ω if $\|\nabla u(x)\| \leq 1$, for a.e. $x \in \Omega$.*
- *u is said to be solution of $\|\nabla u\| = 1$, in Ω satisfying $u = 0$ in C if*

$$u(x) = \max\Big\{ v(x) \ ; \ v \text{ is a.e subsolution of } \|\nabla u\| = 1, \text{ in } \Omega \text{ with } v_{/C} = 0 \Big\}.$$

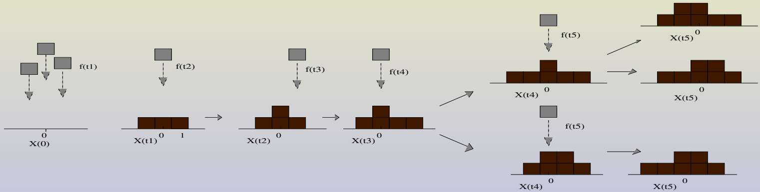
REMARK (equivalent definitions usually used)

- *u is a viscosity subsolution : if $\phi \in \mathcal{C}^1(\Omega)$, $u - \phi$ attains a local maximum at $x \in \Omega$, then $\|\nabla \phi(x)\| \leq 1$.*
- *u is a viscosity supersolution : if $\phi \in \mathcal{C}^1(\Omega)$, $u - \phi$ attains a local maximum at $x \in \Omega$, then $\|\nabla \phi(x)\| \geqslant 1$.*
- *u is a* **viscosity solution** *: u is both a viscosity subsolution and a viscosity supersolution*

Bonus : Microscopic description for the Eikonal equation (a sandpile toy model)

A discrete toy model

The evolution of a stack of unit cubes resting on the plane when new cubes are being added to the pile :



- The cube is assigned on a position connected to several downhill "staircases" along which it can move, and the cube will randomly select among the available downhill paths
- The assigned cube has no "staircases" derived from the position it was put on and remains in place
- A cube moves following adjacent positions in order to get a stable configuration, which means that the heights of any two adjacent columns of cubes can differ by at most one
- In the case of two dimension, a cube moves by falling in one of the four directions (forward, back, left or right)

Consider a set of sites labeled by a couple of integers $i = (i_1, i_2) \in \mathbb{Z}^2$

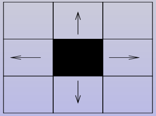
- Let $\hat{C} \subset \hat{\Omega} \subset \mathbb{Z}^2$ be given bounded domains
- The source term is a deterministic function $\hat{f} \ : \ (0, T) \times \hat{\Omega} \to \mathbb{R}$ assigning cubes
- At each time a stable configuration is reached instantaneously ; that is a mapping $\eta(t) \ : \ \hat{\Omega} \to \boldsymbol{N}$ such that

$$\eta = 0 \quad \text{on } \hat{C}$$

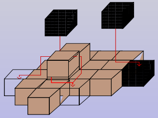
and

$$|\eta(i) - \eta(j)| \leq 1 \quad \text{if} \quad i \sim j,$$

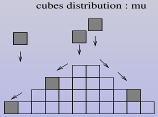
where $i \sim j$ denotes $|i - j| \leq 1$ and $|i| = |i_1| + |i_2|$, for any $i = (i_1, i_2) \in \mathbb{Z}^2$.



$i \sim j \Longleftrightarrow |i - j| \leq 1$



3D



2D

Remark : If we enable the cubes to move in the eight directions by adding the displacements on the diagonal, then we need to equip $\mathbb{Z}^2$ with the norm

$$|(i_1, i_2)| = \max(|i_1|, |i_2|)$$

Particle system for sandpile

- $S = \left\{ \xi \ : \ \hat{\Omega} \to \mathbb{N} \ ; \ \xi_{/\hat{C}} = 0 \text{ and } |\xi(i) - \xi(j)| \leq 1 \text{ for any } i \sim j \right\}$
- $H = l^2(\mathbb{Z}^2)$
- For any $\xi \in S$, we denote by $p(i,j,\xi)$ the probability that a cube at the position i has for to go to the position j :

$$\sum_{j \in \mathbb{Z}^2} p(i,j,\xi) = 1 \text{ pour tout } i \in \mathbb{Z}^2$$

- The infinitesimal generator : for any continuous $F \ : \ B(S) \to \mathbb{R}$ and $\xi \in S$

$$AF(\xi) = \sum_{i,j \in \mathbb{Z}^2} \hat{f}(i,j)\, p(i,j) \Big(F(\xi + \delta_i) - F(\xi) \Big),$$

- **Stochastic Equation :** For any $F \ : \ B(S) \times (0,T) \to \mathbb{R}$ Lipschitz continuous in t, we have

$$F(\eta(t,.),t) - \int_0^t \left(\frac{\partial F}{\partial s} + A\,F \right)(\eta(s,.)) = M(t) \text{ is a Martingale s.t. } E[M(t)] = 0.$$

- Aim : **the associated deterministic model**

$$\lim_{t \to \infty} \lim_{N \to \infty} \boldsymbol{E} \left[\frac{1}{N} \eta([Ns],[Nx]) \right].$$

Fluid limit

Assume that

- $\Omega \subset \mathbb{R}^N$ an open bounded domain and $C \subset \Omega$ a closed domain
- $\hat{f} \equiv 1$.

THEOREM *([Evans-Rezakhanlou,99], [Ig,2008]) We have*

$$\lim_{t\to\infty} \lim_{N\to\infty} \boldsymbol{E}\left[\int_{\mathbb{R}^2} |\frac{1}{N}\,\eta(t,[N\,x]) - u(x)|^2\right] = 0,$$

where u is the unique solution of the Eikonal-equation :

$$\begin{cases} \|\nabla u\| = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

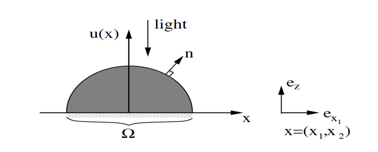
Shape from Shading

- $\Omega \subset \mathcal{R}^2$ be a bounded domain
- $u : \Omega \to \mathcal{R}$ be a function representing the surface of body we want to reconstruct.
- $I : \Omega \to \mathbb{R}^+$ be the brightness of the body (the flux of light per unit of surface).

The traditional SFS problem can be modelled by the so called "*brightness*" or "*image irradiance*" equation with the assumption that :

the material is Lambertian, that is its reflectance is proportional to the scalar product between the normal vector to the surface and the light source direction vector.

$$I(x, y) = \cos(n, e_z)$$



Here n is the unit normal to the surface point $(x, y, u(x, y))$ which can be expressed as

$$n(x, y) = \frac{1}{\sqrt{1 + |\nabla u(x, y)|^2}}(-\partial_x u(x, y), -\partial_y u(x, y), 1).$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$I(x, y) = \frac{1}{\sqrt{1 + |\nabla u(x, y)|^2}} \quad \Leftrightarrow \quad \|\nabla u(x, y)\| = \sqrt{\frac{1}{I(x, y)^2} - 1} =: k(x, y)$$

General theory

Let us consider the Hamilton-Jacobi equation

$$H(x, \nabla u) = 0, \quad \text{in } \Omega$$

where $H : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function (the Hamiltonian) satisfying :

(H1) $Z(x) := \left\{ p \in \mathbb{R}^N \,;\, H(x,p) \leq 0 \right\}$ is **convex,** for any $x \in \Omega$

(H2) $Z(x)$ is compact, for any $x \in \Omega$

(H3) $H(x, 0) \leq 0$, for any $x \in \Omega$.

DEFINITION *Let $u \in \mathcal{C}(\overline{\Omega})$ be locally Lipschitz-continuous and let $y \in \overline{\Omega}$.*

- *u is said to be an a.e. subsolution of $H(x, \nabla u) = 0$, in Ω if $H(x, \nabla u(x)) \leq 0$, for a.e. $x \in \Omega$.*
- *u is said to be solution of $H(x, \nabla u) = 0$, in Ω satisfying $u(y) = 0$ if*

$$u(x) = \max \left\{ v(x) \,;\, v \text{ is a.e subsolution of } H(x, \nabla u) = 0 \text{ in } \Omega \text{ with } v(y) = 0 \right\}.$$

REMARK (equivalent definitions usually used)

- *u is a viscosity subsolution : if $\phi \in \mathcal{C}^1(\Omega)$, $u - \phi$ attains a local maximum at $x \in \Omega$, then $H(x, \nabla \phi(x)) \leq 0$.*
- *u is a viscosity supersolution : if $\phi \in \mathcal{C}^1(\Omega)$, $u - \phi$ attains a local maximum at $x \in \Omega$, then $H(x, \nabla \phi(x)) \leq 0$.*
- *u is a **viscosity solution** : u is both a viscosity subsolution and a viscosity supersolution*

- The family of viscosity solutions is stable with respect the local uniform convergence.
- The pointwise infimum of a family of locally equibounded viscosity solutions is a viscosity solution.
- Let u and v be a viscosity solution and a **strict viscosity subsolution** of $H(x, \nabla u) = 0$ in Ω. If $u = v$ on $\partial\Omega$, then $u \geqslant v$ in Ω.

THEOREM (uniqueness) *Let Ω be an open bounded domain and g a continuous function defined on $\partial\Omega$. If there exists a* **strict subsolution** *of $H(x, \nabla u) = 0$ in Ω, then there is at most one viscosity solution of $H(x, \nabla u) = 0$ in Ω, taking the datum g on the boundary.*

COROLLARY (uniqueness in the supercritical case) *Let Ω be an open bounded domain and g a continuous function defined on $\partial\Omega$. If*

$$(H'3) \qquad H(x, 0) < 0, \qquad \textit{for any } x \in \Omega,$$

then there is at most one viscosity solution of $H(x, \nabla u) = 0$ in Ω, taking the datum g on the boundary.

REMARK *Eikonal equation :*

$$H(x, p) = |p| - k(x).$$

Construction of solutions : metric character of HJ equation

Eikonal equation : u is subsolution of the Eikonal equation in a convex Ω if and only if

$$u(x) - u(y) \le \|x - y\|, \qquad \text{for any } x, y \in \Omega.$$

HJ equation : if u is subsolution of $H(x, \nabla u) = 0$, then

For any $\varphi \in \Gamma(y,x) := \Big\{ \varphi \in \mathrm{Lip}_\Omega;\ \varphi(0) = x \text{ and } \varphi(1) = y \Big\}$, we have

$$u(x) - u(y) = \int_0^1 \nabla u(\varphi(t)) \cdot \varphi'(t)\, dt \le \int_0^1 \sigma(\varphi(t), \varphi'(t))\, dt,$$

where $\sigma(x, .)$ is the support function of $Z(x)$, for any $x \in \Omega$; i.e.

$$\sigma(x, q) = \sup \Big\{ p \cdot q \ ;\ q \in Z(x) \Big\}, \qquad \text{for any } x \in \Omega.$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$u(x) - u(y) \le S(y, x) := \inf \left\{ \int_0^1 \sigma(\xi(s), \xi'(s))\, ds \ ;\ \xi \in \Gamma(y, x) \right\}.$$

REMARK *Eikonal equation :* $\sigma(x, p) = k(x)\, \|p\|$, *for any* $(x; p) \in \Omega \times \mathbb{R}^N$.

Metric character of HJ equation : optical distance (Finsler distance)

$$S(x,y) = \inf\left\{\int_0^1 \sigma(\xi(s),\xi'(s))\,ds\ ;\ \xi \in \Gamma(x,y)\right\},\ \text{for any } x,y \in \Omega.$$

THEOREM

- *S is a quasi-metric ; i.e. S satisfies*
  - $S(x,x) = 0$, *for any* $x \in \Omega$.
  - $S(y,x) \leq S(y,z) + S(z,x)$, *for any* $x,y,z \in \Omega$.
- *For any* $y \in \Omega$, $S(y,.)$ *is a viscosity subsolution in* Ω *and a viscosity supersolution in* $\Omega \setminus \{y\}$ *of* $H(x,\nabla u) = 0$.
- *v is a viscosity subsolution of* $H(x,\nabla u) = 0$ *if and only if*

$$v(x) - v(y) \leq S(y,x), \qquad \text{for any } x,y \in \Omega.$$

- $\delta_g(x) := \min\{g(y) + S(y,x)\ ;\ y \in \partial\Omega\}$ *is the unique viscosity subsolution of* $H(x,\nabla u) = 0$ *satisfying* $u = g$ *on* $\partial\Omega$.

Metric character of HJ equation : optical distance (Finsler distance)

$$S(x,y) = \inf\left\{\int_0^1 \sigma(\xi(s),\xi'(s))\,ds\ ;\ \xi\in\Gamma(x,y)\right\},\ \text{for any } x,y\in\Omega.$$

THEOREM

- *S is a quasi-metric ; i.e. S satisfies*
  - $S(x,x)=0$, *for any* $x\in\Omega$.
  - $S(y,x)\le S(y,z)+S(z,x)$, *for any* $x,y,z\in\Omega$.
- *For any $y\in\Omega$, $S(y,.)$ is a viscosity subsolution in Ω and a viscosity supersolution in $\Omega\setminus\{y\}$ of $H(x,\nabla u)=0$.*
- *v is a viscosity subsolution of $H(x,\nabla u)=0$ if and only if*
  $$v(x)-v(y)\le S(y,x),\qquad \textit{for any } x,y\in\Omega.$$
- *$\delta_g(x):=\min\{g(y)+S(y,x)\ ;\ y\in\partial\Omega\}$ is the unique viscosity subsolution of $H(x,\nabla u)=0$ satisfying $u=g$ on $\partial\Omega$.*

Eikonal equation : $\|\nabla u\|=1$ **in** $\Omega \Longrightarrow \sigma(x,p)=\|p\| \Longrightarrow S(x,y)=\|x-y\|$,

- u is a subsolution of $\|\nabla u\|=1$ if and only if $u(x)-u(y)\le\|y-x\|$.

  For a fixed $a\in\Omega$, the function $u\ :\ x\in\Omega\to u(x):=\|a-x\|$ is a viscosity solution of $\|\nabla u\|=1$ in $\Omega\setminus\{a\}$, satisfying $u(a)=0$.

  The function $d(x,\partial\Omega)$ is the viscosity solution of $\|\nabla u\|=1$ in Ω, satisfying $u=0$ on $\partial\Omega$.

New approach : constrained optimization problem

HJ equation VS Maximization volume problem

$$\begin{cases} H(x, \nabla u) = 0, & \text{in } \Omega \setminus C \\ u = 0 & \text{on } C \end{cases}$$

(H1) $Z(x) := \left\{ p \in \mathbb{R}^N \, ; \, H(x,p) \leq 0 \right\}$ is **convex,** for any $x \in \Omega$

(H2) $Z(x)$ is compact, for any $x \in \Omega$

(H3) $H(x, 0) \leq 0$, for any $x \in \Omega$.

Theorem *(cf. [Ig&al,2017], [Ig&al,2019]) Under the assumptions (H1-H3), the viscosity solution is given by*

$$\begin{aligned} \int u \, dx &= \max\left\{ \int z \, dx \, : \, z \in Lip(\Omega), \, z_{/C} = 0, \, z(x) - z(y) \leq S(y,x) \right\}. \\ &= \max\left\{ \int z \, dx \, : \, z \in Lip(\Omega), \, z_{/C} = 0, \, \sigma^*(x, \nabla z) \leq 1 \right\} \\ &= \min\left\{ \int \sigma(x, \frac{\Phi}{|\Phi|}) \, d|\Phi| \, : \, \Phi \in \mathcal{M}_b(\Omega)^N, \, -\nabla \cdot \Phi = 1 \ in \ \mathcal{D}'(\Omega \setminus C) \right\}. \end{aligned}$$

Moreover, if Φ is optimal then

$$\begin{cases} \left. \begin{array}{c} -\nabla \cdot \Phi = 1 \\ \Phi \cdot \nabla u = \sigma(x, \Phi) \end{array} \right\} & in \ \Omega \setminus C \\ \Phi \cdot n = 0 & on \ \partial(\Omega \setminus C) \end{cases}$$

The case of Eikonnal equation

$$\begin{cases} \|\nabla u\| = k, & \text{in } \Omega \setminus C \\ u = 0 & \text{on } C \end{cases} \qquad \left| \begin{array}{l} k \in \mathcal{C}(\overline{\Omega}) \\ k \geqslant 0 \end{array} \right.$$

Then

- $\sigma(x, y) = k(x) \, \|p\|$ for any $(x, p) \in \Omega \times \mathbb{R}^N$
- $d(x, y) = \inf \left\{ \int_0^1 k(\xi(s)) \, \|\xi'(s)\| \, ds \; ; \; \xi \in \Gamma(x, y) \right\}$, for any $x, y \in \Omega$

COROLLARY *The viscosity solution is given by*

$$\begin{aligned} \int u \, dx &= \max \left\{ \int z \, dx \; : \; z \in Lip(\Omega), \; z_{/C} = 0, \; z(x) - z(y) \leq d(y, x) \right\}. \\ &= \max \left\{ \int z \, dx \; : \; z \in Lip(\Omega), \; z_{/C} = 0, \; \|\nabla z\| \leq 1 \right\} \\ &= \min \left\{ \int k(x) \, d|\Phi|(x) \; : \; \Phi \in \mathcal{M}_b(\Omega)^N, \; -\nabla \cdot \Phi = 1 \text{ in } \mathcal{D}'(\Omega \setminus C) \right\}. \end{aligned}$$

Moreover, if Φ is optimal then

$$\begin{cases} \left. \begin{array}{l} -\nabla \cdot \Phi = 1 \\ \Phi \cdot \nabla u = k \, |\Phi| \end{array} \right\} & \text{in } \Omega \setminus C \\ \Phi \cdot n = 0 & \text{on } \partial(\Omega \setminus C) \end{cases} \qquad \Leftrightarrow$$

The case of Eikonnal equation

$$\begin{cases} \|\nabla u\| = k, & \text{in } \Omega \setminus C \\ u = 0 & \text{on } C \end{cases} \qquad \left| \begin{array}{l} k \in \mathcal{C}(\overline{\Omega}) \\ k \geqslant 0 \end{array} \right.$$

Then

- $\sigma(x, y) = k(x)\, \|p\|$ for any $(x, p) \in \Omega \times \mathbb{R}^N$
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Moreover, if Φ is optimal then

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Numerical algorithm

Related works

$$\begin{cases} \|\nabla u\| = & k \quad in \quad \Omega \\ \\ u = & 0 \quad on \quad C \subset \partial\Omega. \end{cases} \quad \left| \begin{array}{l} k \in \mathcal{C}(\overline{\Omega}) \\ \\ k > 0 \end{array} \right. \tag{5}$$

- Optimal control (cf. [Camilli&al,]) :
  $$\dot{v}(t) = p(t) \text{ for } \; t \in [0, \infty), \quad v(0) = x,$$
  where p is a measurable function and they introduce a cost functional
  $$J(x, p) = \int_0^T k(v(t)) \, \|p(t)\| \, dt.$$

- Fast Marching Method (FMM) and Fast Sweeping Method (FSM) :
  The FSM is based on an upwind difference discretization solved via Gauss-Seidel iterations with alternating sweeping ordering.

- Elliptic approach : (cf. [Glowinsky&al,]) : $k \equiv 1$

$$\min \left\{ J(v) := \int_\Omega |\nabla v|^2 dx - C \int_\Omega v dx \; : \; v \in H^1(\Omega) : v \text{ solves } (5) \right\}$$

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$$\min \left\{ \tilde{J}(v) := J(v) + \frac{\epsilon_1}{2} \int_\Omega |\Delta v|^2 dx + \frac{1}{4\epsilon_2} \int_\Omega (|\nabla v|^2 - 1)^2 dx : v \in H^2(\Omega), v_{/C} = 0 \right\}.$$

Duality approach

$$\begin{cases} H(x, \nabla u) = 0, & \text{in } \Omega \setminus C \\ u = 0 & \text{on } C \subseteq \partial\Omega \end{cases}$$

$$\begin{aligned} \int u\, dx &= \max\left\{ \int z\, dx \; : \; z \in Lip(\Omega),\; z_{/C} = 0,\; \sigma^*(x, \nabla z) \le 1 \right\} \\ &= \min\left\{ \int \sigma(x, \frac{\Phi}{|\Phi|})\, d|\Phi| \; : \; \Phi \in \mathcal{M}_b(\Omega)^N,\; -\nabla \cdot \Phi = 1 \text{ in } \mathcal{D}'(\Omega \setminus C) \right\}. \end{aligned}$$

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Fenchel-Rockafellar duality

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The Augmented Lagrangian method

Fenchel-Rockafellar duality approach

Let us consider

- two Banach spaces $\mathscr{X}, \mathscr{Y}$
- two convex l.s.c functions $\mathcal{F} : \mathscr{X} \to (\infty, +\infty]$ and $\mathcal{G} : \mathscr{Y} \to (\infty, +\infty]$.
- Primal problem

$$(\mathcal{P}) \quad \inf_{u \in \mathscr{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u)$$

  with $\Lambda \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$.
- Dual problem, which is given by

$$(\mathcal{D}) \quad \sup_{v \in \mathscr{Y}} \Big(-\mathcal{F}^*(-\Lambda^* v) - \mathcal{G}^*(v) \Big)$$

  Here Λ^* is the adjoint operator of Λ, while $\mathcal{F}^*, \mathcal{G}^*$ are the Legendre-Fenchel transformations of $\mathcal{F}$ and $\mathcal{G}$ given by

$$\mathcal{F}^*(f) = \sup_{u \in \mathscr{X}} (\langle f, u \rangle - \mathcal{F}(u)) \quad \mathrm{f} \in \mathscr{X}^*$$

$$\mathcal{G}^*(g) = \sup_{q \in \mathscr{Y}} (\langle g, q \rangle - \mathcal{G}(q)) \quad \mathrm{g} \in \mathscr{Y}^*$$

  with $\mathscr{X}^*, \mathscr{Y}^*$ are respectively the dual space of $\mathscr{Y}$ and $\mathscr{Y}$.
- Weak duality :

$$\sup_{v \in \mathscr{Y}} \Big(-\mathcal{F}^*(-\Lambda^* v) - \mathcal{G}^*(v) \Big) \leq \inf_{u \in \mathscr{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u)$$

- Strong duality :

$$\sup_{v \in \mathscr{Y}} \Big(-\mathcal{F}^*(-\Lambda^* v) - \mathcal{G}^*(v) \Big) = \inf_{u \in \mathscr{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u)$$

Alg2

We introduce a new primal variable $q \in \mathscr{Y}$ and we write $(\mathcal{P})$ in the following alternative form

$$(\tilde{\mathcal{P}}) \quad \inf_{\substack{(u,q)\in \mathscr{X}\times\mathscr{Y} \\ \Lambda u = q}} \mathcal{F}(u) + \mathcal{G}(q).$$

So solving $(\tilde{\mathcal{P}})$ consists in finding a saddle point to the following Augmented Lagrangian

$$L_r(u,q,\phi) = \mathcal{F}(u) + \mathcal{G}(q) + \langle \phi, \Lambda u - q\rangle + \frac{r}{2}|\Lambda u - q|^2$$

with $r > 0$. This means solving

$$(\mathscr{S}) : \min_{(u,q)\in\mathscr{X}\times\mathscr{Y}} \max_{\phi\in\mathscr{Y}^*} L_r(u,q,\phi).$$

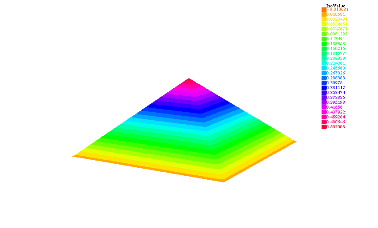
We initialize with ϕ_0, q_0 and the algorithm consists in optimizing alternatively in u, q, ϕ.

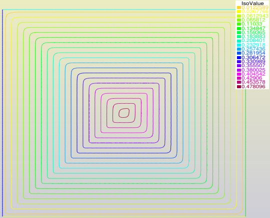
- **1st step :** $u_{i+1} \in argmin_{u\in\mathscr{X}} \left\{ \mathcal{F}(u) + \langle \phi_i, \Lambda(u)\rangle + \frac{r}{2}|\Lambda(u) - q|^2 \right\}.$
- **2nd step :** $q_{i+1} \in argmin_{w\in\mathscr{Y}} \left\{ \mathcal{G}(q) - \langle \phi_i, q\rangle + \frac{r}{2}|\Lambda(u_{i+1}) - q|^2 \right\}.$
- **3rd step :** We update the multiplier ϕ using a step size equal to the Lagrangian parameter r :

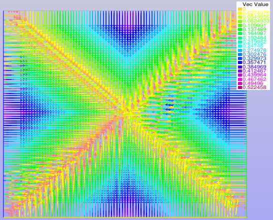
$$\phi_{i+1} = \phi_i + r(\Lambda u_{i+1} - q_{i+1}).$$

Numerical results

The case $|\nabla u| = 1$ in $\Omega = (0,1)^2$ and $u = 0$ on the boundary.







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Merci pour votre attention

"Le vrai voyageur ne sait pas où il va"
"The real traveler does not know where he is going."
Marcel Proust, 1871-1922, French writer.