New results on maximal L^p-regularity for evolution equations in Banach spaces

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Outline

1. The concept of Maximal L^p-regularity
2. Lutz Weis's results on Maximal Regularity in UMD spaces
3. New results on maximal L^p-regularity: perturbation approach
4. Application to Volerra integro-differential equations

Definition of Maximal Regularity for autonomous equations

Definition

Let X Banach, $1 < p < \infty$. We say that $A : D(A) \subset X \to X$ has the maximal L^p–regularity on $[0,\tau]$ ($A \in MR_p(0,\tau;X)$) if

$$\forall f \in L^p(0,\tau;X), \quad \exists! u(\cdot) \in W^{1,p}(0,\tau;X) \cap L^p(0,\tau;D(A))$$

satisfying The evolution equation

$$\dot{u}(t) = Au(t) + f(t), \quad t \in [0,\tau], \qquad u(0) = 0$$

- If $A \in MR_p(0,\tau;X)$ then there exists $c > 0$ such that
$$\int_0^\tau \|\dot{u}(t)\|^p dt + \int_0^\tau \|Au(t)\|^p dt \le C \int_0^\tau \|f(t)\|^p dt.$$
- "Maximal" means that the better regularity of $\dot{u}$ and Au is L^p ($\dot{u} - Au = f$).

First conditions for maximal L^p-regularity

Necessary condition (Dore Veni)

If $A \in MR_p(0,\tau;X)$ and X **Banach**, then $A : D(A) \subset X \to X$ is the generator of holomorphic semigroup on X. In particular

$$\sup_{\lambda \in \Sigma_\sigma} \|\lambda R(\lambda, A)\| < \infty$$

for some $\sigma > \frac{\pi}{2}$, where $R(\lambda, A) := (\lambda I - A)^{-1}$ and the sector

$$\Sigma_\sigma := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg(\lambda)| < \sigma\}$$

NSC on Hilbert spaces (Dore Veni)

If X is a **Hilbert** space, then $A \in MR_p(0,\tau;X)$ iff A is the generator of holomorphic semigroup on X.

The role of maximal L^p-regularity

Maximal L^p-regularity helps in proving well-posedess of e.g.

- Non-autonomous evolution equations
- Non-linear equations
- quasi-linear evolution equations

Illustration to an example:Non-autonomous evolution equations

Consider the non-autonomous Cauchy problem

$$\dot{u}(t) = A(t)u(t) + f(t), \qquad u(0) = x \qquad \text{(nCP)}$$

Where $A(t) : D \subset X \to X$, $t \in [0, \tau]$ is a family of analytic generators such that

$$\|R(\lambda, A(t))\| \leq c\,(1 + |\lambda|)^{-1}, \qquad \lambda \in \mathbb{C}^+.$$

We assume that $A(\cdot) : [0, \tau] \to \mathcal{L}(D, X)$ is continuous

For example

$$A(t) = \sum_{i,j=1}^{n} a_{ij}(t, \cdot) \frac{\partial^2}{\partial x_i \partial x_j}$$

Illustration to an example:Non-autonomous evolution equations

For $f = 0$, we need also

$$(s \mapsto u(s) = e^{sA(t)}x) \in W^{1,p}([0,\tau], X) \cap L^p([0,\tau], D).$$

This is the case if $x \in Y$, an interpolation space between D and X.

For $x \in Y$ we reformulate (nCP) as

$$\dot{u}(t) = A(0)u(t) + g_u(t), \qquad u(0) = x,$$

where

$$g_u(t) = [A(t) - A(0)]u(t) + f(t).$$

Illustration to an example:Non-autonomous evolution equations

Denote by

$$[L(u)](t) := e^{A(0)t}x + \int_0^t e^{(t-s)A(0)} g_u(s)ds$$

The solution of the autonomous equation

$$\dot{u}(t) = A(0)u(t) + g_u(t), \qquad u(0) = x,$$

Then, at least formally, the solutions of (nCP) are the fixed point of the map L.

Maximal regularity will now used to show that L is a contraction in $L^p([0,a], D)$ for a small $a \leq \tau$.

Illustration to an example:Non-autonomous evolution equations

If u_1, u_2 in $L^p([0,a], D)$, then $L(u_1) - L(u_2)$ equals the solution of

$$\dot{u}(t) = A(0)u(t) + (g_{u_1}(t) - g_{u_2}(t)), \qquad u(0) = 0,$$

Maximal regularity now implies that

$$\begin{aligned} \|L(u_1) - L(u_2)\|_{L^p([0,a],D)} &\leq \|A(0)u\|_{L^p([0,a],X)} \\ &\leq C\|g_{u_1} - g_{u_2}\| \\ &= \|[A(\cdot) - A(0)](u_1 - u_2)\|_{L^p([0,a],X)} \\ &\leq \gamma_a \|u_1 - u_2\|_{L^p([0,a],D)} \end{aligned}$$

where

$$\gamma_a := C \sup_{s\in[0,a]} \|A(t) - A(0)\|_{\mathcal{L}(D,X)}.$$

Unconditional Martingale Differences spaces: UMD

A Banach space X is called UMD-space if the Hilbert transform

$$(\mathcal{H}f)(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|s|>\epsilon} \frac{f(t-s)}{s} ds, \quad t \in \mathbb{R}, \quad f \in \mathcal{S}(\mathbb{R}, X),$$

is extend to a bounded operator on $L^p(\mathbb{R}, X)$.

- UMD spaces are reflexive
- Any Hilbert space is UMD
- the L^p-spaces $p \in (1, \infty)$ are UMD

Maximal L^p-regularity in UMD spaces

$\tau \subset \mathcal{L}(X, Y)$ is $\mathcal{R}$-bounded if $\exists\, C > 0$ s.t. for all $n \in \mathbb{N}$, $T_1, ..., T_n \in \tau$, $x_1, ..., x_n \in X$, $r_j : [0, 1] \to \{-1, 1\}$ independent random variables,

$$\int_0^1 \| \sum_{j=1}^{n} r_j(s) T_j x_j \|_Y ds \le C \int_0^1 \| \sum_{j=1}^{n} r_j(s) x_j \|_X ds$$

Weis's Theorem in UMD spaces

Let A generates of a bd analytic semigroup in a UMD-space X. Then

$A \in MR_p(0, \tau; X) \iff \{\lambda R(\lambda, A) : \lambda \in \rho(A)\}$ is $\mathcal{R}$ – bounded.

Weis's theorem in Banach spaces

We assume that A generates an analytic semigroup $T(t))_{t\geq 0}$ on a **Banach** space X.

We select

$$(R_A f)(t) := A\int_0^t T(t-s)f(s)ds, \qquad f \in C(0,\tau;D(A))$$

Weis's theorem says that $A \in MR_p(0,T;X)$ iff there exists $C > 0$ such that

$$\|R_A f\|_p \leq C\|f\|_p, \qquad \forall f \in f \in C(0,\tau;D(A)).$$

Maximal regularity under small perturbations

- A closed operator $A : D(A) \subset X \to X$ is **sectorial** if $(0, \infty) \subset \rho(A)$ and

$$\sup_{t>0} \|t(t+A)^{-1}\| < \infty.$$

- A linear operator $P : D(P) \subset X \to X$ is small for A if $D(A) \subset D(P)$ and for every $\delta > 0$, there exists $c_\delta > 0$ such that

$$\|Px\| \leq \delta\|Ax\| + c_\delta\|x\|, \quad \forall x \in D(A).$$

Theorem (Kunstmann & Weis 2001)

Let A be sectorial in a **UMD**-space X, $A \in MR_p(0, \tau; X)$ and P be a small perturbation for A. Then $A + P \in MR_p(0, \tau; X)$.

More perturbations: admissible perturbations

$P \in \mathcal{L}(D(A), X)$ is said to be p-admissible perturbation for A if for some (hence all) $\alpha > 0$ there exists $\gamma := \gamma(\alpha) > 0$ such that

$$\int_0^\alpha \|PT(t)x\|^p dt \leq \gamma^p \|x\|^p, \qquad \forall x \in D(A).$$

Examples: Let A generates a bd analytic semigroup.

- If $\theta \in (0, \frac{1}{p})$ then $P := (-A)^\theta$ is p-admissible for A.

  The proof uses the estimate

  $$\|t^\theta A^\theta T(t)\| \leq M, \qquad \forall t \geq 0.$$

- If $\theta \in (\frac{1}{p}, 1)$ then $P := (-A)^\theta$ is never p-admissible for A.

Analyticity of the perturbed semigroup

X Banach, and $P \in \mathcal{L}(D(A), X)$ p-admissible

Theorem

Assume A generates an analytic semigroup $(T(t))_{t\geq 0}$. Then $(A^P := A + P, D(A))$ generates an analytic semigroup T^p on X

P p-admissible implies there exists $\alpha_0 > \omega_0(A)$ such that $\mathbb{C}_{\alpha_0} \subset \rho(A^P)$ and

$$\|R(\lambda, A^P)\| \leq 2\|R(\lambda, A)\|, \quad \forall \lambda \in \mathbb{C}_{\alpha_0}.$$

As T analytic, then T^P is analytic.

Amansag, Bounit, Driouich, S.H, J. Evolution Equations 2019

A perturbation results

X Banach, and $P \in \mathcal{L}(D(A), X)$ p-admissible, $f \in L^p_{loc}(\mathbb{R}^+, X)$,

$$\dot{u}(t) = (A+P)u(t) + f(t), \quad u(0) = 0. \qquad \text{(CP)}$$

Theorem (H., Semigroup Forum 2005)

There exists an extension $\tilde{P} : D(\tilde{P}) \subset X \to X$ of P such that the solution of (CP) satisfies

$$u(t) \in D(\tilde{P}) \quad \text{for } a.e.\ t > 0,$$

$$\|\tilde{P}u(\cdot)\|_{L^p([0,\alpha],X)} \le c_\alpha \|f\|_{L^p([0,\alpha],X)},$$

$$u(t) = \int_0^t T(t-s)[\tilde{P}u(s) + f(s)]ds$$

Maximal L^p-regularity under admissible perturbations

X Banach, and $P \in \mathcal{L}(D(A), X)$ p-admissible

Theorem (Amansag, Bounit, Driouich, H., 2019)

$$A \in MR_p(0,\tau;X) \iff A^P = A + P \in MR_p(0,\tau;X)$$

For any $f \in C([0,\tau], D(A))$ we define

$$(\mathcal{R}f)(t) = A\int_0^t T(t-s)f(s)ds, \quad (\mathcal{R}^P f)(t) = A^P \int_0^t T^P(t-s)f(s)ds.$$

We have

$$\mathcal{R}^P f = \mathcal{R}g, \quad g = \tilde{P}u(\cdot) + f, \qquad \|g\|_p \le \kappa \|f\|_p$$

If $A \in MR_p(0,\tau;X)$, then

$$\|\mathcal{R}g\|_p \le C\|g\|_p \le C\kappa\|f\|_p.$$

An example: a heat equation in a no reflexive state space

Take $\alpha \in [0, \frac{1}{p})$ and Consider the heat equation

$$\dot{z}(t) = \Delta z(t) + (-\Delta)^\alpha z(t) + f(t), \qquad z(0) = 0$$

on the **Besov space** $X := \dot{B}^0_{1,p}(\mathbb{R}^n)$ which is no reflexive.

Ogawa & Shimizu (2010) proved that $\Delta \in MR_p(\dot{B}^0_{1,p}(\mathbb{R}^n))$.

We know that $(-\Delta)^\alpha$ is p-admissible.

Thus $\Delta + (-\Delta)^\alpha \in MR_p(\dot{B}^0_{1,p}(\mathbb{R}^n))$.

Application to Volterra operator

Let X be a Banach space and introduce the space

$$\mathcal{X} = X \times L^q(\mathbb{R}^+, X), \qquad \left\|\left(\begin{smallmatrix} x \\ g \end{smallmatrix}\right)\right\| = \|x\| + \|g\|_q,$$

and the matrix operator

$$\mathfrak{A} = \begin{pmatrix} A & \delta_0 \\ a(\cdot)F & \frac{d}{ds} \end{pmatrix}, \quad D(\mathfrak{A}) = D(A) \times W^{1,q}(\mathbb{R}^+, X).$$

Where

- $A : D(A) \subset X \to X$ generates a C_0-sg $(T(t))_{t\geq 0}$ on X,
- $a : \mathbb{R}^+ \to \mathbb{C}$ and $F : D(A) \to X$ a linear operator

Problem

Assume that $A \in MR_p(0, \tau; X)$. Does $\mathfrak{A}$ generates an analytic semigroup and $\mathfrak{A} \in MR_p(0, \tau; \mathcal{X})$ for some $p \in (1, \infty)$?

Application to Volterra operator

We split

$$\mathfrak{A} = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{ds} \end{pmatrix} + \begin{pmatrix} 0 & \delta_0 \\ a(\cdot)F & 0 \end{pmatrix}$$
$$:= \mathfrak{A}_0 + \mathcal{P}.$$

Clearly $\mathfrak{A}_0$ generates the following semigroup on $\mathcal{X}$,

$$\mathcal{T}_0(t) = \begin{pmatrix} T(t) & 0 \\ 0 & S(t) \end{pmatrix}, \qquad t \geq 0,$$

where $(S(t))_{t\geq 0}$ is the right shift semigroup $L^q(\mathbb{R}^+, X)$:

$$(S(t)g)(s) = g(t+s), \qquad t, s \geq 0.$$

Observe that $(\mathcal{T}_0(t))_{t\geq 0}$ is not analytic in $\mathcal{X} = X \times L^q(\mathbb{R}^+, X)$.

The use of Bergman space

Take $\theta \in (0, \frac{\pi}{2})$ and define

$$\Sigma_\theta := \{z \in \mathbb{C} : \ |\arg(z)| < \theta\}$$

The Bergman space

$$B^q_{\theta,X} := \left\{ f : \Sigma_\theta \to X : \text{ holomorphic } \int_{\Sigma_\theta} \|f(\sigma + ir)\|^q_X d\sigma dr < \infty \right\}.$$

$$\|f\|_{B^q_{\theta,X}} := \left(\int_{\Sigma_\theta} \|f(\sigma + ir)\|^q_X d\sigma dr \right)^{\frac{1}{q}}.$$

Application to Volterra operator

On $B^q_{h,X}$, we define the complex derivative $\frac{d}{dz}$ with domain

$$D\left(\frac{d}{dz}\right) := \left\{f \in B^q_{h,X}; f' \in B^q_{h,X}\right\}.$$

$\left(\frac{d}{dz}, D\left(\frac{d}{dz}\right)\right)$ generates an analytic semigroup on $B^q_{\theta,X}$.

If X is UMD then $\frac{d}{dz} \in MR_p(0,\tau; B^q_{\theta,X})$.

If X is UMD and $A \in MR_p(0,\tau;X)$ then

$$\mathfrak{A}_0 = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{dz} \end{pmatrix} \in MR_p(0,\tau;\mathcal{X}^q), \quad \mathcal{X}^q := X \times B^q_{\theta,X}.$$

Application to Volterra operator

Theorem (Amansag, Bounit, Driouich, H., 2019)

Let X UMD space, $s \in (1,2)$, $q = \frac{ps}{s-1}$, $A \in MR_p(0,\tau;X)$, $a \in B^q_{\theta,\mathbb{C}}$ and $F \in \mathcal{L}(D(A),X)$ p-admissible for A. Then

$$\mathfrak{A} = \begin{pmatrix} A & \delta_0 \\ a(\cdot)F & \frac{d}{ds} \end{pmatrix} = \mathfrak{A}_0 + \mathcal{P} \in MR_p(0,\tau; X \times B^q_{\theta,X}).$$

We prove that

$$\int_0^\alpha \|f(t)\|_X^p dt \le c_\alpha \|f\|^p_{B^q_{\theta,X}}.$$

This implies that

$$\int_0^\alpha \|\mathcal{P}\mathcal{T}_0(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\|^p_{\mathcal{X}^q} \le \gamma \left(\|x\| + \|f\|^p_{B^q_{\theta,X}} \right)^p.$$

where $\gamma = h(\|a\|_{B^q_{\theta,\mathbb{C}}}, c_\alpha)$.

Unbounded perturbations of generator domain

Let X, U and Z such that $Z \hookrightarrow X$, $A_m : D(A_m) = Z \to X$ differential operator, $G, M : Z \to U$.
Consider the boundary value problem

$$(BVP) \quad \begin{cases} \dot{w}(t) = A_m w(t) + f(t), & t \geq 0, \qquad w(0) = 0, \\ Gw(t) = Mw(t), & t \geq 0. \end{cases}$$

We assume

- $G : Z \to U$ is surjective
- $A := A_m$ with $D(A) = \{x \in Z : Gx = 0\}$ is a generator of C_0–semigroup on X

The Dirichlet operator

$$D_\lambda := \left(G_{|\ker(\lambda - A_m)}\right)^{-1}, \qquad \lambda \in \rho(A).$$

Unbounded perturbations of generator domain

If we define

$$\mathcal{A} := A_m, \quad D(\mathcal{A}) := \{x \in Z : Gx = Mx\}$$

then (BVP) is equivalent to

$$\dot{(w)}(t) = \mathcal{A}w(t) + f(t), \quad w(0) = 0, \qquad t \in [0, \tau]. \qquad \text{(CP)}$$

In H., Manzo, Rhandi, Disct. conti. Dyn. System A (2015), we proved there exists an extension $(\tilde{M}, D(\tilde{M}))$ of M such that

$$\mathcal{A} := A + (\lambda - A)D_\lambda \tilde{M}.$$

Maximal L^p-regularity for $\mathcal{A}$ is obtained in
Amansag, Bounit, Driouich and H., SIAM J. Math. Anal. to appear

Thank You