Fractional Laplacian

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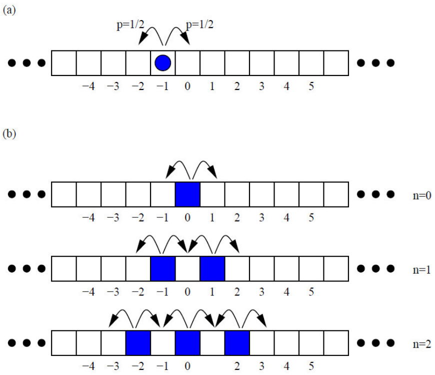
Laplacian

Discrete random walk



State:
Let $x_i = i\triangle x$ with $i \in \mathbb{Z}$ be the location of the particle at time $t_n = n\triangle t$.

Dynamics:
If the particle is in state x_i at time step t_n, it will jump either to x_{i-1} or to x_{i+1} with equal probabilities.

Discrete random walk

Define

$p(m, n) =$ probability that the particle is in state x_m at time step t_n.

REMARK

$$p(m, n) = \left(\frac{1}{2}\right)^n \binom{n}{a} = \left(\frac{1}{2}\right)^n \frac{n!}{a!(n-a)!} \quad \text{where} \quad a = \frac{n+m}{2}.$$

Discrete random walk and the heat equation

Master equation

$$p(m;n) = \frac{1}{2}p\Big(m-1, n-1\Big) + \frac{1}{2}p\Big(m+1; n-1\Big).$$

Discrete random walk and the heat equation

Master equation

$$p(m;n) = \frac{1}{2}p\Big(m-1,n-1\Big) + \frac{1}{2}p\Big(m+1;n-1\Big).$$

We now scale the master equation, using

$$\triangle x \to 0, \qquad \triangle t \to 0, \qquad \frac{\triangle x^2}{2\triangle t} = D.$$

We assume that the scaled probabilities $p(m,n)$ approach a continuous (and even twice differentiable!) function $u(x;t)$

$$u(x;t) = p\left(\frac{x}{\triangle x}, \frac{t}{\triangle t}\right)$$

Discrete random walk and the heat equation

Then

$$u(x;t) = \frac{1}{2}u(x - \triangle x, t - \triangle t) + \frac{1}{2}u(x + \triangle x, t - \triangle t),$$

Discrete random walk and the heat equation

Then

$$u(x;t) = \frac{1}{2}u(x - \triangle x, t - \triangle t) + \frac{1}{2}u(x + \triangle x, t - \triangle t),$$

or equivalently,

$$\frac{u(x;t) - u(x, t - \triangle t)}{\triangle t}$$
$$= \frac{\triangle x^2}{2\triangle t}\ \frac{u(x - \triangle x, t - \triangle t) - 2u(x, t - \triangle t) + u(x + \triangle x, t - \triangle t)}{\triangle x^2}.$$

In the limit $\triangle x \to 0$, $\triangle t \to 0$, and $\frac{\triangle x^2}{2\triangle t} = D$, we obtain the heat equation

$$u_t = Du_{xx}.$$

Discrete random walk and the heat equation

In the case of the random walk on the d-dimensional lattice $(\triangle x)\mathbb{Z}^d$, we obtain

$$u_t = D\Delta u \quad \left(= D \sum_{k=1}^{d} \frac{\partial^2 u}{\partial x_k^2} \right).$$

Discrete random walk and the heat equation

In the case of the random walk on the d-dimensional lattice $(\triangle x)\mathbb{Z}^d$, we obtain

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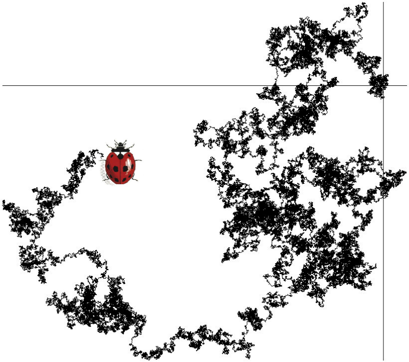
Fundamental solution

$$\mathcal{N}(x,t) = \frac{1}{(2\pi Dt)^{n/2}} \exp\left(\frac{-|x|^2}{4Dt} \right)$$

Laplace operator & Wiener process



Brownian motion – one trajectory of a Wiener process

Laplace operator & Wiener process

Definition
The stochastic process $\{W(t)\}_{t\geq 0}$ is called the Wiener process, if it fulfils the following conditions

- $W(0) = 0$ with probability equal to one,
- $W(t)$ has independent increments ,
- trajectories of W are continuous with probability equal to one
- $\forall_{0\leq s\leq t}\ W_t - W_s \sim \mathcal{N}(0, t-s)$.

For every function $u_0 \in C_b(\mathbb{R}^n)$ we define

$$u(x,t) = E^x(u_0(W(t))) = \int_{\mathbb{R}^n} u_0(x-y)\,\mathcal{N}(0,t)(dy),$$

where $\mathcal{N}(0,t)(dy) = (2\pi t)^{-n/2} e^{-|y|^2/(2t)} dy$.
Hence

$$u_t = \frac{1}{2}\Delta u \quad \text{oraz} \quad u(x,0) = u_0(x).$$

Fractional Laplacian

Random walk and fractional Laplacian

Let $\Pi : \mathbb{R}^d \to [0, +\infty)$ satisfies

$$\Pi(y) = \Pi(-y) \quad \text{for any} \quad y \in \mathbb{R}^d,$$

and

$$\sum_{k \in \mathbb{Z}^d} \Pi(k) = 1.$$

New notation: $h = \triangle x, \quad \tau = \triangle t$

Give a small $h > 0$, we consider a **random walk** on the lattice $h\mathbb{Z}^d$.

Dynamics

- at any unit of time τ, a particle jumps from any point of $h\mathbb{Z}^d$ to any other point;
- the probability for which a particle jumps from the point $hk \in h\mathbb{Z}^d$ to the point $h\tilde{k}$ is taken to be $\Pi(k - \tilde{k}) = \Pi(\tilde{k} - k)$.

Random walk and fractional Laplacian

We call $u(x, t)$ the probability that our particle

$$\text{lies at} \quad x \in h\mathbb{Z}^d \quad \text{at time} \quad t \in \mathbb{Z}.$$

Random walk and fractional Laplacian

We call $u(x,t)$ the probability that our particle

$$\text{lies at} \quad x \in h\mathbb{Z}^d \quad \text{at time} \quad t \in \mathbb{Z}.$$

$$u(x,t+\tau) = \sum_{k\in\mathbb{Z}^d} \Pi(k) u(x+hk,t).$$

Hence,

$$u(x,t+\tau) - u(x,t) = \sum_{k\in\mathbb{Z}^d} \Pi(k)\big(u(x+hk,t) - u(x,t)\big).$$

Random walk and fractional Laplacian

Particularly nice asymptotics are obtained in the case

$$\tau = h^\alpha \qquad \text{and} \qquad \Pi(y) = \frac{C}{|y|^{d+\alpha}} \text{ for } y \neq 0$$

and $\Pi(0) = 0$.

We observe that

$$\frac{\Pi(k)}{\tau} = h^d \Pi(hk).$$

Random walk and fractional Laplacian

Hence

$$\begin{aligned}\frac{u(x,t+\tau)-u(x,t)}{\tau} &= \sum_{k\in\mathbb{Z}^d}\frac{\Pi(k)}{\tau}\big(u(x+hk,t)-u(x,t)\big)\\ &= h^d\sum_{k\in\mathbb{Z}^d}\Pi(hk)\big(u(x+hk,t)-u(x,t)\big)\\ &= h^d\sum_{k\in\mathbb{Z}^d}\psi(hk,x,t).\end{aligned}$$

where

$$\psi(y,x,t)=\Pi(y)\big(u(x+y,t)-u(x,t)\big).$$

Random walk and fractional Laplacian

Notice that

$$h^d \sum_{k \in \mathbb{Z}^d} \psi(hk, x, t) \to \int_{\mathbb{R}^d} \psi(y, x, t)\, dy \quad \text{when} \quad h \to 0.$$

Consequently, passing to the limit $\tau = h^\alpha \to 0$ we obtain the equation

$$u_t(x, t) = \int_{\mathbb{R}^d} \psi(y, x, t)\, dy,$$

that is

$$u_t(x, t) = C \int_{\mathbb{R}^d} \frac{u(x + y, t) - u(x, t)}{|y|^{n+\alpha}}\, dy.$$

NOTATION

$$u_t(x, t) = -(-\Delta)^{\alpha/2} u(x, t)$$

Fractional Laplacian

Now, we compute the Fourier transform of the equation

$$u_t(x,t) = C \int_{\mathbb{R}^d} \frac{u(x+y,t) - u(x,t)}{|y|^{n+\alpha}}\, dy.$$

to obtain

$$\widehat{u}_t(\xi,t) = C(\alpha,n)|\xi|^{\alpha}\widehat{u}(\xi,t)$$

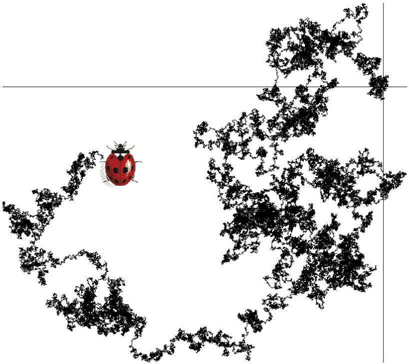
where

$$C(\alpha,n) = C \int_{\mathbb{R}^d} \frac{e^{i\xi_0 y} - 1}{|y|^{n+\alpha}}\, dy < 0.$$

Fractional Laplacian

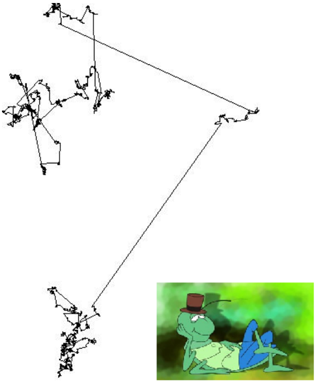
$$\big(\widehat{(-\Delta)^{\alpha/2}} v\big)(\xi) = |\xi|^{\alpha}\widehat{v}(\xi).$$

Laplace operator & Wiener process



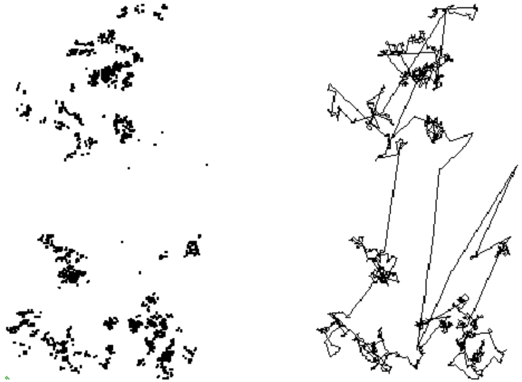
Brownian motion – one trajectory of a Wiener process

Lévy process



One trajectory of a Lévy process

Lévy process



Two pictures of the same trajectory of a Lévy process

Lévy process

Definition
The stochastic process $\{X(t) \ : \ t \geq 0\}$ on the probability space (Ω, F, P) is called the Lévy process with values in $\mathbb{R}^n$ if it fulfils the following conditions:

- X(0) $= 0$, P-p.w.,
- for every sequence $0 \leq t_0 < t_1 < \cdots < t_n$ random variables $X(t_0), X(t_1) - X(t_0), \dots, X(t_n) - X(t_{n-1})$ are independent,
- the law of $X(s+t) - X(s)$ is independent of s,
- the process $X(t)$ is continuous in probability, namely, $\lim_{s \to t} P(|X_s - X_t| > \varepsilon) = 0$.

Lévy-Khinchin formula

Lévy operator:

$$\mathcal{L}u(x) \quad = \quad b \cdot \nabla u(x) - \sum_{j,k=1}^{d} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} - \int_{\mathbb{R}^d} \Big(u(x-\eta) - u(x) \Big) \, \Pi(d\eta),$$

where

- $b \in \mathbb{R}^d$ is a given vector,
- $(a_{jk})_{j,k=1}^{d}$ is a given nonnegative definite matrix
- Π in a Borel measure satisfying $\Pi(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min(1, |\eta|^2) \, \Pi(d\eta) < \infty$$

Fractional Laplacian

Let

$$\Pi(d\eta) = \frac{C(\alpha)}{|\eta|^{n+\alpha}} \quad \text{with} \quad \alpha \in (0, 2)$$

in

$$\mathcal{L}u(x) = -\int_{\mathbb{R}^d} \Big(u(x-\eta) - u(x) \Big) \, \Pi(d\eta).$$

We obtain the α-stable anomalous diffusion equation:

$$u_t + (-\Delta)^{\alpha/2} u = 0$$

Fundamental solution of the equation $u_t + (-\Delta)^{\alpha/2} u = 0$

Define the function $p_\alpha(x,t)$ by the Fourier transform:

$$\widehat{p}_\alpha(\xi,t) = e^{-t|\xi|^\alpha}. \quad \text{Note that} \quad p_2(x,t) = (4\pi t)^{-d/2} e^{-|x|^2/(4t)}.$$

- Scaling:

$$p_\alpha(x,t) = t^{-d/\alpha} P_\alpha(x t^{-1/\alpha}), \quad \text{where} \quad (P_\alpha)\check{}(\xi) = e^{-|\xi|^\alpha}.$$

- For every $\alpha \in (0,2)$, the function P_α is smooth, nonnegative, $\int_{\mathbb{R}^d} P_\alpha(x)\, dx = 1$, and satisfies

$$0 \le P_\alpha(x) \le C(1+|x|)^{-(\alpha+d)} \quad \text{and} \quad |\nabla P_\alpha(x)| \le C(1+|x|)^{-(\alpha+d+1)}$$

for a constant C and all $x \in \mathbb{R}^d$.

Maximum principle

Maximum principle

Definition
The operator A satisfies the **positive maximum principle** if for any $\varphi \in D(A)$ the fact

$$0 \leq \varphi(x_0) = \sup_{x \in \mathbb{R}^n} \varphi(x) \quad \text{for some} \quad x_0 \in \mathbb{R}^n$$

implies

$$A\varphi(x_0) \leq 0.$$

□

REMARK
$A\varphi = \varphi''$ or, more generally, $A\varphi = \Delta\varphi$ satisfies the positive maximum principle.

Maximum principle

THEOREM
Denote by $\mathcal{L}$ the Lévy diffusion operator. Then $A = -\mathcal{L}$ satisfies the positive maximum principle.

Proof
Assume that $0 \leq \varphi(x_0) = \sup_{x \in \mathbb{R}^n} \varphi(x)$. Then

$$\begin{aligned} -\mathcal{L}\varphi(x_0) &= -b \cdot \nabla\varphi(x_0) + \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 \varphi(x_0)}{\partial x_j \partial x_k} \\ &\quad + \int_{\mathbb{R}^n} \Big(\varphi(x_0 - \eta) - \varphi(x_0) \Big) \Pi(d\eta) \leq 0. \end{aligned}$$

□

Convexity inequality

THEOREM
Let $u \in C_b^2(\mathbb{R}^n)$ and $g \in C^2(\mathbb{R})$ be a convex function. Then

$$\mathcal{L}g(u) \leq g'(u)\mathcal{L}u.$$

Proof. Use the representation

$$\mathcal{L}u(x) \quad = \quad b \cdot \nabla u(x) - \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} - \int_{\mathbb{R}^n} \Big(u(x-\eta) - u(x) \Big) \Pi(d\eta).$$

and the convexity of g

$$g(u(x-\eta)) - g(u(x)) \geq g'(u(x))[u(x-\eta) - u(x)].$$

□

Nonlinear models with fractional Laplacian

Fractal Burgers equation

$$u_t + (-\Delta)^{\alpha/2} u + u u_x = 0$$

$$\text{where} \quad x \in \mathbb{R}.$$

Self-interacting individuals

Differential equations describing the behavior of a collection of self-interacting individuals via pairwise potentials arise in the modeling of animal collective behavior: ocks, schools or swarms formed by insects, fishes and birds.

The simplest model:

$$\frac{dx_j}{dt} = -\sum_{j \neq i} m_j \nabla K(x_i - x_j).$$

Here, x_i is the position of the particle with mass m_i.

The continuum descriptions

$$u_t = -\nabla \cdot \big(u(\nabla K * u)\big).$$

Here, the unknown function $u = u(x,t) \geq 0$ is either the population density of a species or the density of particles in a granular media.

Model of chemotaxis

$$u_t = -(-\Delta)^{\alpha/2} u - \nabla \cdot \big(u(\nabla K * u) \big)$$

where $\qquad \alpha \in (0, 2].$