Lec. 1:

7 p. $\sim$ 2h. $\div$ 10 min pause

Plan:

today { - Intro
- Visc. sol'n

Tuesd. $\to$ - Comp. pr. lpf

Wednesday - Numerics

A. Non-loc. op's

$(x \in \mathbb{R}^N,\ z \in \mathbb{R}^M)$

$$(L) \quad L\varphi(x) = \int_{\mathbb{R}^M} \{\varphi(x + j(x,z)) - \varphi(x) - j(x,z)\cdot D\varphi(x)\}\,\mu(dz)$$

(μ) μ Borel meas. on $\mathbb{R}^M$, $\mu \geq 0$, $\mu(\{0\}) = 0$, $\int_{\mathbb{R}^M} |z|^2 \mu(dz) < \infty$

(j) $|j(x,z)| \leq C|z|$, $|j(x,z) - j(y,z)| \leq L_j |z||x-y|$

Ex. 1:

a) $\mu(dz) = \dfrac{c_\alpha\, dz}{|z|^{N+\alpha}}$, $\alpha \in (0,2)$, $j \equiv z \Longrightarrow L \approx -(-\Delta)^{\frac{\alpha}{2}}$ $(M=N)$

Fractional Laplace

b) $\mu(dz) = \begin{cases} C\dfrac{e^{-Gz}\,dz}{|z|^{1+Y}}, & z > 0 \\ C\dfrac{e^{+Mz}\,dz}{|z|^{1+Y}}, & z < 0 \end{cases}$; $C, G, M \geq 0$, $Y \in (0,2)$; $j = x(e^z - 1)$

$j(x,z) = x(e^z - 1)$

CGMY-model $\to$ Finance $\to$ "Many other models"

$(M = N = 1)$

c) $\mu = \delta_1$, $j = z$ Poisson proc. ; d) $\mu(dz) = g(z)\,dz$, $0 \leq g \in L^1$; $j = z$ compound Poisson

Rem. 1:

i) L generator of SDE w. jumps.

ii) More generally: $L\varphi = \int_{|z|<1} \{\varphi(x+j) - \varphi - j\cdot D\varphi\}\mu + \int_{|z|>1} \{\varphi(x+j) - \varphi(x)\}\mu$

$\hookrightarrow$ $-(-\Delta)^{\frac{\alpha}{2}}$ ~~Ex. 1a)~~ of this type ((μ), (j) hold for $|z|<1$!)

Obs: $|\varphi(x+j) - \varphi(x) - j\cdot D\varphi(x)| \overset{\text{Taylor}}{=} \left|\frac{1}{2} j^T D^2\varphi(\xi) j\right| \overset{(j)}{\leq} \frac{1}{2} C^2 |D^2\varphi(\xi)| \cdot |z|^2$,

where $|\xi - x| \leq |j| \overset{(j)}{\leq} C|z|$

Hence

$$|L\varphi(x)| = \left|\int_{|z|<1} \cdots\right| + \left|\int_{|z|>1} \cdots\right|$$

$$\leq \tfrac{1}{2} C^2 \|D^2\varphi\|_{L^\infty(B(x,\varepsilon))} \int_{|z|<1} |z|^2 \mu(dz)$$

$$+ 2\|\varphi\|_{L^\infty} \underbrace{\int_{|z|>1} \mu(dz)}_{\leq \int |z|^2\mu} + |D\varphi(x)| \underbrace{\int_{|z|>1} |z|\,\mu(dz)}_{\leq \int |z|^2 \mu}$$

$\Downarrow$ (μ)

Lem. 1: $\varphi \in C^2 \cap L^\infty \Rightarrow L\varphi(x)$ well-def.

B. Non-loc. eq'ns

(1) $F(x, u, Du, D^2u, Lu) = 0$ in Ω

(F1) $F = F(x, u, p, X, l)$ cont.; non-decreasing in $\cancel{X}$ l; decreasing in u

Opt. control, diff. games, front propagation, $\infty-\Delta$, MCM, HJ-eq'ns, math. finance, obstacle problems, degenerate lin. eq'ns ...

Here - simplify:

(2) $u + F(x, Du, Lu) = 0$ in $\mathbb{R}^N$

Ex. 2: $u + |Du| - Lu = f(x)$ in $\mathbb{R}^N$

(3) $u + \max_{\alpha \in \mathcal{A}} \{-L^\alpha u + b^\alpha(x) \cdot Du + f^\alpha(x)\} = 0$ in $\mathbb{R}^N$

Rem. 2 p. 4,5

Degenerate elliptic problems in general (e.g. in opt. ctrl.)

$\Rightarrow$ Sol'ns not smooth in general

$\Rightarrow$ need "weak" sol'ns

Problems:

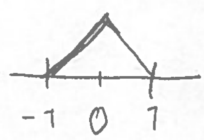
a) Non-divergence form $\leadsto$ can not (easily) use distr. sol'ns and i.b.p.

b) "Strong" (a.e.) sol'ns are not unique.

Ex. 3:

$$|u_x| = 1 \text{ in } (-1,1), \quad u(\pm 1) = 0$$

"Sol'n": $u(x) = 1 - |x| = \mathrm{dist}(x, \partial(-1,1))$



u not differentiable at $x = 0$!

u "strong" (a.e.) sol'n, but so are

$u_1 =$  , $u_2 =$  , ...

10 min pau

C. Viscosity sol'ns

Correct sol'n concept for (1) and (2)!

$\leadsto$ uniqueness, existence ++

Lem. 2: $\psi \in C^2$ has a global max. at x_0

$$\Rightarrow D\psi(x_0) = 0, \quad D^2\psi(x_0) \le 0, \quad L\psi(x_0) \le 0$$

$$\Big[\int \{ \underbrace{\psi(x_0+j) - \psi(x_0)}_{\le 0} - j \cdot \underbrace{D\psi(x_0)}_{=0} \} \underbrace{\mu(dz)}_{\ge 0} \le 0 \Big]$$

$u - \varphi$ glob. max. in x_0

Lem. 2
$\Rightarrow Du(x_0) = D\varphi(x_0)$, $D^2u(x_0) \le D^2\varphi(x_0)$, $Lu(x_0) \le L\varphi(x_0)$

u solve (I)
$\Rightarrow 0 = F(x_0, u(x_0), Du(x_0), D^2u(x_0), Lu(x_0))$

$$\overset{(FI)}{\geq} F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0), L\varphi(x_0))$$

Prop. 1 p. 4,5 (*)

i) and ii) makes sense for u cont.,
(or even u u.s.c. in i) and l.s.c. in ii)) !

Rem. 2:

i) f u.s.c. (l.s.c.) at x if $\limsup_{y \to x} f(y) \le f(x)$
$\left(\liminf_{y\to x} f(y) \geq f(x)\right)$

iii) u.s.c. + l.s.c. $\Leftrightarrow$ cont. [easy - chk!]

iv) f u.s.c. (l.s.c.) $\Rightarrow$ f attains its max (min) on every compact set.

Def. 1:

i) An bnd u.s.c. func'n u is a visc. subsol'n of (I) if for every $\varphi \in C^2 \cap L^\infty$ and global max. pt. $x_0 \in \Omega$ of $u - \varphi$,

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0), L\varphi(x_0)) \le 0$$

ii) - l.s.c. ——— visc. supersol'n ———
——— min. pt. ———

$$F(\text{- - - -}) \geq 0$$

iii) A cont. func'n u is visc. sol'n if it is a visc. sub and supersol'n.

p. 4

Prop. 1: $u \in C^2 \cap L^\infty$; (F1) holds.

u cl. sol'n of (1)

$$\Updownarrow$$

$$\begin{cases} \text{i) } \forall \varphi \in C^2 \cap L^\infty, x_0 \in \Omega \text{ s.t. } u-\varphi \text{ has glob. max in } x_0, \\ \qquad F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0), L\varphi(x_0)) \le 0. \\ \text{ii) } \forall \varphi \in C^2 \cap L^\infty, x_0 \in \Omega \text{ s.t. } u-\varphi \text{ has glob. min in } x_0, \\ \qquad F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0), L\varphi(x_0)) \ge 0. \end{cases}$$

Pf.: $\Downarrow$) done, see (*) ...

$\Uparrow$) Take $u = \varphi$. $u - \varphi \equiv 0$ has glob. max/min in every pt.

$$\overset{\text{i) / ii)}}{\Longrightarrow} \quad F(x, u(x), Du(x), \dots) = F(x, u(x), D\varphi(x), \dots) \overset{\text{i)/ii)}}{\underset{\ge}{\le}} 0 \text{ in } \Omega$$

$\square$

p. 4

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p. 2

Rem. 2: Opt. stoch. ctrl.

$$\begin{cases} dX_t = b^{\alpha_t}(X_t)\,dt + \int_{|z|>0} j^{\alpha_t}(X_t, z)\, \overbrace{\tilde{N}(dz, dt)}^{\text{Poisson rnd. meas.}}, \; X_0 = x \\ \qquad\qquad\qquad\qquad \text{jump-term} \\ u(x) = \inf_{\alpha_t \in \mathcal{C}} \left\{ E \int_0^\infty \underbrace{e^{-t} f^{\alpha_t}(X_t)}_{\text{discounted running cost}} dt \right\} \end{cases}$$

$[\varphi^\alpha(x) = \varphi(x, \alpha)]$

$E\int_0^1 \int_B \tilde{N}(dz, dt) = \underset{\text{intensity}}{\mu(B)}$ $\forall$ Borel B

$$\Updownarrow$$

$u(x)$ visc. sol'n of (3) (= HJB eq'n)

p. 3

p. 3

<u>Rem. 3</u>: By Prop. 1:

i) A classiacal sol'n u is a visc. sol'n

ii) A $C^2 \cap L^\infty$ visc. sol'n is a cl. sol'n.

<u>Pf.</u>:

a) By $(*)$, u is a visc. subsol'n. Supersol'n is similar Lem. 3

b) Take $\varphi = u$, $u - \varphi (\equiv 0)$ glob. max / min. in every pt.

$$\overset{u \text{ v. sol'n}}{\Longrightarrow} F(x, u(x), Du(x), \dots) = F(x, u(x), D\varphi(x), \dots) \lessgtr 0 \text{ in } \Omega$$

$\square$

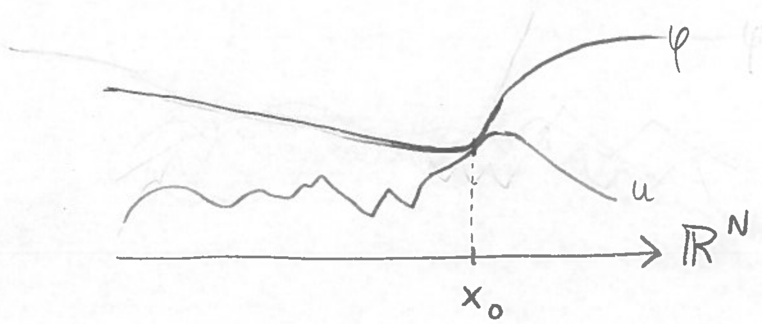
<u>Rem. 4</u>:

i) May assume $u(x_0) = \varphi(x_0)$ in def. 1

[replace $\varphi(x)$ by $\varphi(x) - u(x_0)$ ...]

ii) $u - \varphi$ glob. max. (min.) at x_0 + i)

$\Rightarrow$ φ touches u from above (below)



$u \leq \varphi$

$u(x_0) = \varphi(x_0)$

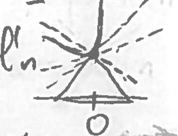
<u>Ex. 3</u> (cont.)

$u(x) = 1 - |x|$ is visc. sol'n of $|u_x| = 1$ in $(-1,1)$:

$x \neq 0 \Rightarrow u$ smooth cl. sol'n $\overset{\text{Prop. 1 b}}{\Longrightarrow}$ u visc. sol'n

$x = 0$: $u - \varphi$ max in x_0 $\Rightarrow$ $|\varphi_x(x_0)| \leq 1$ (chk) $\Rightarrow u$ subsol'n

$u - \varphi$ min in x_0 $\to$ impossible $\Rightarrow u$ supersol'n



<u>Chk.</u>: The other str. sol'ns are not visc. sol'ns! You can take φ s.t. x_0 min., $\varphi_x(x_0) \neq 0$ $\Rightarrow$ $|\varphi_x(x_0)| \neq 0$,

D. Alternative definition

Needed for uniqueness! (Other good for stability)

$L\varphi = L_r\varphi + L^r[\varphi, D\varphi]$

$L_r\varphi(x) = \int_{|z|<r} \cdots\, \mu(dz)$

$L^r[u,p](x) = \int_{|z|>r} \{u(x+j(x,z)) - u(x) - p\cdot j(x,z)\}\, \mu(dz)$

As in Lem. 1:

$\begin{matrix} \varphi \in C^2 \\ u \text{ bnd.} \end{matrix} \Rightarrow \begin{matrix} L_r\varphi(x) \text{ well-def.} \\ L^r[u, D\varphi(x)](x) \text{ — " —} \end{matrix}$

$u - \varphi$ glob. max in x_0

"Lem. 2" $\Rightarrow$ $L_r u(x_0) \le L_r\varphi(x_0)$, $L^r[u,p](x_0) \le L^r[\varphi, p](x_0)$

Def. 2:

i) A bnd u.s.c. func'n u is a visc. sub.sol'n of (1) if for every $\underbrace{\varphi}_{r>0,} \in C^2$ and glob. max pt. $x_0 \in \Omega$ of $u - \varphi$,

$\qquad F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0), L_r\varphi(x_0) + L^r[u, D\varphi(x_0)](x_0)) \le 0$

ii) — super sol'n — iii) — sol'n —

Prop. 2: Assume (F1): Def 1 $\Leftrightarrow$ Def 2.

Pf.:

$2 \Rightarrow 1$: $\quad 0 \ge F(x_0, \ldots, L_r\varphi(x_0) + L^r[u, D\varphi](x_0)) \overset{(F1)}{\ge} F(x_0, \ldots, L\varphi(x_0))$

$L = L_r + L^r$

$L^r[u, D\varphi](x_0) \le L^r[\varphi, D\varphi](x_0)$

"$1 \Rightarrow 2$:" Take smooth u_ε s.t. $u \le u_\varepsilon \le \varphi$, $u_\varepsilon \to u$ a.e. (!)

$u - \varphi$ max in $x_0 \Rightarrow u - u_\varepsilon$ and $u_\varepsilon - \varphi$ max in x_0

$\Rightarrow 0 \ge F(x_0, \ldots, Lu_\varepsilon(x_0)) = F(\ldots, \underbrace{L_r u_\varepsilon(x_0)}_{\le L_r\varphi(x_0)} + \underbrace{L^r[u_\varepsilon, \overset{D\varphi(x_0)}{\overset{\shortparallel}{Du_\varepsilon}}](x_0)}_{\to L^r[u, D\varphi](x_0)})$ $\square$

Lec. 2: 1h 15 min? 4,5 p. 1,5h! Repeter: Def. 2 + $L = L_r + L^r$ Slides el. på tavle f. foreslsn.

E. Comparison and uniqueness

(2) $u + F(x, Du, Lu) = 0$ in $\mathbb{R}^N$.

(L) $Lu(x) = \int_{\mathbb{R}^M} \{u(x + j(x,z)) - u(x) - j(x,z)Du(x)\}\mu(dz)$

(μ) $\mu \geq 0$ Borel meas., $\mu(\{0\}) = 0$, $\int_{\mathbb{R}^M} |z|^2 \mu(dz) < \infty$

(j) $|j(x,z)| \leq C|z|$, $|j(x,z) - j(y,z)| \leq L_j |x-y| \cdot |z|$

(F1) $F = F(x,p,l)$ cont., $l_1 \leq l_2 \Rightarrow F(x,p,l_1) \geq F(x,p,l_2)$

(F2) $F(x,p,l_1) - F(y,q,l_2) \leq C\{(1+|p|+|q|)|x-y| + |p-q| + (l_1 - l_2)^-\}$ non-char in l

Ex. 4, p. 7,5

Thm. 1: (Comp.) $\forall$ u visc. subsol'n, v visc. supersol'n of (2)

$$\Rightarrow u \leq v \text{ in } \mathbb{R}^N.$$

Cor. 1: Visc. sol'n of (2) is unique.

[2 sol'ns $u_1 \neq u_2 \overset{\text{Thm. 1}}{\Longrightarrow} u_1 \leq u_2 \leq u_1$, contradiction]

Pf. by doubling of variables: "Crandall, Lions, Evans in local case, Soner in non-loc. case. Kruzkov..." Brakke

$\Phi(x,y) := u(x) - v(y) - \phi(x,y)$

$\phi(x,y) := \frac{1}{\varepsilon}\varphi(x-y) + \psi(\delta x) + \psi(\delta y)$; $\varepsilon, \delta > 0$; $\varphi, \psi \geq 0$, symm.

$C^2 \ni \varphi(z) = \begin{cases} |z|^2, & |z| < 1 \\ \geq 0, & 1 \leq |z| < 2 \\ 2, & |z| \geq 2 \end{cases}$, $C^2 \ni \psi(r) = \begin{cases} 0, & |z| < 1 \\ 2(\|u\|_\infty + \|v\|_\infty) + 1, & |z| > 2 \end{cases}$

p.7

Ex.: 4: (Chk.!)

$$F(x,p,l) = \sup_{\alpha\in\mathcal{A}}\{b^\alpha(x)\cdot p + f^\alpha(x)\} + G(l)$$

satisfy (F2) (and (F1)) if $\exists\ M, L, \Lambda > 0$ s.t.

$$|b^\alpha(x)| + |f^\alpha(x)| \le M$$

$$|b^\alpha(x) - b^\alpha(y)| + |f^\alpha(x) - f^\alpha(y)| \le L|x-y|$$

$$G \in C^1; \qquad -\Lambda \le G'(l) \le 0$$

for all x, α, l.

not unif. ell.!

$(\cancel{G'(l) \le -\lambda} < 0)$

Rem. 5: More gen. assumptions ⇝ literature!

p.7

$$M := \sup_{\mathbb{R}^N \times \mathbb{R}^N} \Phi(x,y)$$

Lem. 3:

$\exists (\bar{x}, \bar{y})$ s.t. $\Phi(\bar{x},\bar{y}) = M$ and

ii) $\varphi(\bar{x}-\bar{y}) \le 2\varepsilon (\|u\|_\infty + \|v\|_\infty)$

iii) $\lim\limits_{\varepsilon \to 0} \frac{1}{\varepsilon} \varphi(\bar{x}-\bar{y}) = 0$ (δ fixed!)

i) $\boxed{|\bar{x}| + |\bar{y}| \le \frac{2}{\delta}}$

iv) $\lim\limits_{\delta \to 0} \lim\limits_{\varepsilon \to 0} (\psi(\delta\bar{x}) + \psi(\delta\bar{y})) = 0$

Pf.:

i) $\Phi(0,0) = u(0) - v(0) \ge -(\|u\|_\infty + \|v\|_\infty) \overset{\text{def. }\psi}{\ge} -2(\|u\|_\infty + \|v\|_\infty) - 1$

$|x|$, or $|y| > \frac{2}{\delta} \overset{\text{chk!}}{\Longrightarrow} \Phi(x,y) \le u(x) - v(y) \underbrace{- \psi(\delta x) - \psi(\delta y)}_{\le -(\|u\|_\infty + \|v\|_\infty + 1)}$

$\overset{\text{comp.}}{\Longrightarrow} \exists (\bar{x},\bar{y})$ and $|\bar{x}|, |\bar{y}| < \frac{2}{\delta}$

ii) $\Phi(\bar{x},\bar{x}) + \Phi(\bar{y},\bar{y}) \le 2\Phi(\bar{x},\bar{y})$ ...

iii) $\bar{x}, \bar{y} \to \tilde{x}$ as $\varepsilon \to 0$ along a subseq'nce by i) and ii) and comp.

$m := \sup_{\mathbb{R}^N} \Phi(x,x) \le \Phi(\bar{x},\bar{y})$

$\Longrightarrow \frac{1}{\varepsilon}\varphi(\bar{x}-\bar{y}) \le u(\bar{x}) - v(\bar{y}) - \psi(\delta\bar{x}) - \psi(\delta\bar{y}) - m$

iv) Drop... $\overset{-v,u \text{ u.s.c.}}{\Longrightarrow} \limsup\limits_{\varepsilon \to 0} \frac{1}{\varepsilon}\varphi(\bar{x}-\bar{y}) \le \underbrace{u(\tilde{x}) - v(\tilde{x}) - 2\psi(\delta\tilde{x})}_{=\Phi(\tilde{x},\tilde{x}) \le m} - m \le 0$ $\square$

Lem. 4:

$D_x\phi(\bar{x},\bar{y}) = \frac{1}{\varepsilon} D\varphi(\bar{x}-\bar{y}) + o_\delta(1)$ as $\delta \to 0$

$D_y\phi(\bar{x},\bar{y}) = -\frac{1}{\varepsilon} D\varphi(\bar{x}-\bar{y}) + o_\delta(1)$ "

$L\phi(\cdot,\bar{y})(\bar{x}) = \frac{1}{\varepsilon} L\varphi(\cdot - \bar{y})(\bar{x}) + o_\delta(1)$ "

$L\phi(\bar{x},\cdot)(\bar{y}) = \frac{1}{\varepsilon} L\varphi(\bar{x} - \cdot)(\bar{y}) + o_\delta(1)$ "

where $o_\delta(1)$ does not depend on ε.

Pf.:

$D^k(\psi(\delta x)) = \delta^k (D^k\psi)(\delta x)$; $D\psi, D^2\psi$ bnd; "Lem. 1" + Lem 3 iv) ... $\square$

Lem. 5: $\varphi \in C^2 \Rightarrow L_r\varphi(x) \to 0$ as $r \to 0$

$$\left[L_r\varphi(x) \le \text{"Lem. 7"} \le \frac{1}{2} c^2 \|D^2\varphi\|_{L^\infty(B(x,c))} \int |z|^2 \mathbb{1}_{|z|<r}\, \mu(dz) \xrightarrow[r\to 0]{\text{LDCT}+(\mu)} 0 \right]$$

$\mathbb{1}_r = \begin{cases} 1 & |z|<r \\ 0 & |z|\ge r \end{cases}$

Pf. of Thm. 1:

1) Since $\Phi(x,\bar{y}) \le \Phi(\bar{x},\bar{y}) = M \quad \forall x$ (Lem. 3),

$$u(x) - \phi(x,\bar{y}) \le u(\bar{x}) - \phi(\bar{x},\bar{y}) \qquad (\text{max in } \bar{x})$$

and since u sub sol'n (Def. 2),

$$u(\bar{x}) + F(\bar{x}, \underbrace{D_x\phi(\bar{x},\bar{y})}_{p_x}, \underbrace{L_r\phi(\cdot,\bar{y})(\bar{x}) + L[u, D_x\phi(\bar{x},\bar{y})](\bar{x})}_{l_x}) \le 0$$

2) Similarly $\Phi(\bar{x},y) \le \Phi(\bar{x},\bar{y})$ and

$$-v(y) - \phi(\bar{x},y) \le -v(\bar{y}) - \phi(\bar{x},\bar{y}).$$

Hence $v - (-\phi)$ has glob. min. at $\bar{y}$ and (supersol'n)

$$v(\bar{y}) + F(\bar{y}, \underbrace{D_y(-\phi)(\bar{x},\bar{y})}_{p_y}, \underbrace{L_r(-\phi)(\bar{x},\cdot)(\bar{y}) + L[v, D_y(-\phi)(\bar{x},\bar{y})](\bar{y})}_{l_y}) \ge 0$$

3) By (F2),

$$u(\bar{x}) - v(\bar{y}) \le F(\bar{y},\ldots) - F(\bar{x},\ldots)$$

$$\le C\Big\{ \underbrace{(1+|p_x|+|p_y|)|\bar{x}-\bar{y}| + |p_x - p_y|}_{A} + \underbrace{(l_y - l_x)^-}_{B} \Big\}$$

4) A: By Lem. 3, if ε small enough,

$$\frac{1}{\varepsilon}\varphi(\bar{x}-\bar{y}) = \frac{1}{\varepsilon}|\bar{x}-\bar{y}|^2, \quad \frac{1}{\varepsilon}(D\varphi)(\bar{x}-\bar{y}) = 2\left(\frac{\bar{x}-\bar{y}}{\varepsilon}\right).$$

Then since $D_x(\varphi(x-y)) = D_y((-\varphi)(x-y))$ and Lem 4,

OBS! $\longrightarrow$ $$|p_x - p_y| = \left|\cancel{\tfrac{1}{\varepsilon}(D\varphi)(\bar{x}-\bar{y})} + \sigma_\delta(1) - \cancel{\tfrac{1}{\varepsilon}(D\varphi)(\bar{x}-\bar{y})} + \sigma_\delta(1)\right| = \sigma_\delta(1)$$

and hence

$$A \le C \underbrace{\frac{|\bar{x}-\bar{y}|^2}{\varepsilon}}_{=\frac{1}{\varepsilon}\varphi(\bar{x}-\bar{y})} + \sigma_\delta(1) \overset{\text{Lem. 3 iii)}}{\le} \underbrace{\sigma_\varepsilon(1)}_{\text{dep. on }\delta} + \sigma_\delta(1)$$

5) B: $(l_y - l_x)^- = (l_x - l_y)^+$ and

$$l_x - l_y = \underbrace{L_r^{\ulcorner}\phi(\cdot,\bar{y})(\bar{x}) - L_r\phi(\bar{x},\cdot)(\bar{y})}_{C} + \underbrace{L^{\ulcorner}[u, D_x\phi(\bar{x},\bar{y})](\bar{x}) - L^{\ulcorner}[v, D_y(-\phi)(\bar{x},\bar{y})](\bar{y})}_{D}$$

By Lem. 5,

$$C = \frac{1}{\varepsilon}\sigma_r(1) \qquad (\text{as } r \to 0)$$

Since (Soner)

$$\Phi(\bar{x} + \underbrace{j(\bar{x},z)}_{j_x}, \bar{y} + \underbrace{j(\bar{y},z)}_{j_y}) \le \Phi(\bar{x},\bar{y}),$$

$$u(\bar{x}+j_x) - u(\bar{x}) - (v(\bar{y}+j_y) - v(\bar{y})) \le \phi(\bar{x}+j_x, \bar{y}+j_y) - \phi(\bar{x},\bar{y})$$

Hence by integration ((L)),

$$D \le \int_{|z|>r} \{\phi(\bar{x}+j_x, \bar{y}+j_y) - \phi(\bar{x},\bar{y}) - D_x\phi(\bar{x},\bar{y})j_x + D_y(-\phi)(\bar{x},\bar{y})j_y\}\mu(dz)$$

$$\overset{\text{"Lem. 4"}}{=} \frac{1}{\varepsilon}\int_{|z|>r} \{\varphi(\bar{x}-\bar{y}+j_x-j_y) - \varphi(\bar{x}-\bar{y}) - (D\varphi)(\bar{x}-\bar{y})\cdot(j_x-j_y)\}\mu(dz)$$

chk!

$$+ \sigma_\delta(1)$$

$$\overset{\text{Taylor}}{\le} \|D^2\varphi\|_\infty \frac{1}{\varepsilon}\int_{|z|>r} |j_x - j_y|^2 \mu(dz)$$

$$\overset{(j)}{\le} \|D^2\varphi\|_\infty c^2 \underbrace{\frac{|\bar{x}-\bar{y}|^2}{\varepsilon}}_{=\frac{1}{\varepsilon}\varphi(\bar{x}-\bar{y})} \int_{|z|>0} |z|^2\mu(dz) \overset{\text{Lem. 3 iii)}}{=} \sigma_\varepsilon(1)$$

$|j_x - j_y| \le c|z||\bar{x}-\bar{y}|$ $\qquad$ (μ)

6) Conclusion: For any $x \in \mathbb{R}^N$

$$u(x) - v(x) - 2\varphi(\delta x) = \Phi(x,x) \le \Phi(\bar{x},\bar{y}) = u(\bar{x}) - v(\bar{y}) - \phi(\bar{x},\bar{y})$$

$$\underset{\text{Lem. 3}}{\overset{1)-5)}{\le}} \frac{1}{\varepsilon}\sigma_r(1) + \underbrace{\sigma_\varepsilon(1)}_{\text{dep. on }\delta} + \sigma_\delta(1)$$

Send $r \to 0$, then $\varepsilon \to 0$, and finally $\delta \to 0$:

$$u(x) - v(x) \le 0.$$

□

Lec. 3: Numerics

A. Non-local op.

(L) $L\varphi(x) = \int \{ \varphi(x + j(x,z)) - \varphi(x) - j(x,z) \cdot D\varphi(x) \} \mu(dz)$

(μ) $\mu \geq 0$ Borel meas., $\mu(\{0\}) = 0$, $\int |z|^2 \mu(dz) < \infty$

(j) $|j(x,z)| \leq c|z|$, $|j(x,z) - j(y,z)| \leq L|x-y|$

Simplify:

(S) $j \equiv z$; $\mu(dz) = \mu(z)dz$, $\mu(z) = \mu(-z)$

$L^1_{loc}(\mathbb{R}^N \setminus \{0\})$

abuse of notation!

Lem. 1:

(L), (μ), (S)

$$\Rightarrow L\varphi(x) = \underbrace{\frac{1}{2}\int_{\mathbb{R}^N} \{ \varphi(x+z) - 2\varphi(x) + \varphi(x-z) \} \mu(dz)}_{\text{conv.!}}$$

B. Discretize L

comp. supp meas.

a) Finite meas. μ^r: $\mu \rightsquigarrow \mu^r := \mathbb{1}_{|z|>r}\mu$; $L \rightsquigarrow L^r$

$$\int_{\mathbb{R}^N} \mu^r(z)dz = \int_{|z|>r} \mu(z)dz \leq \int_{|z|>r} \left|\frac{z}{r}\right|^2 \mu(z)dz \overset{(\mu)}{<} \infty$$

$$|L^r\varphi(x) - L\varphi(x)| = \Big|\underbrace{\int_{|z|<r} \cdots}_{L_r\varphi(x)}\Big| \xrightarrow[r \to 0]{\varphi \in C^2 \cap L^\infty} 0 \text{ by Lem. 5}$$

B. Discretization of L

$$L_h\varphi(x) := \frac{1}{2}\sum_{|i|\le P_h}\{\varphi(x+z_i) - 2\varphi(x) + \varphi(x-z_i)\}\mu_i \quad , \quad h>0$$

(A1) Conv.: $|L_h\varphi(x) - L\varphi(x)| \longrightarrow 0$ as $h\to 0$ $\forall \varphi\in C^2\cap L^\infty$

(A2) Positive: $\mu_i \ge 0 \quad \forall i$

(A3) Equi.dist.: $z_i = h\cdot i \in h\mathbb{Z}^N$

(A4) Comp. stencil: $P_h < \infty \quad (\in \mathbb{N})$

~~Obs. 1:~~ ~~$\mu_i \ge 0 \ \forall i \Rightarrow L_h\varphi(x_0) \le 0$ at glob. max. pt's of φ~~

Lem. 2: (A2) + $\varphi\in C_b$ has glob. max. at x_0

$$\Longrightarrow L_h\varphi(x_0) \le 0$$

C. Discretization of eq'n

Simplify:

(3) $\quad u + F(Lu) = f(x)$

(F1) $F\in C^1$, $-L_F \le F'(l) \le 0$, $f\in C_b$

Grid: $\mathcal{G}_h = h\mathbb{Z}^N = \{x_i\}_i = \{hi\}_i \quad (i \in \mathbb{Z}^N)$

Approx.: $U : \mathcal{G}_h \to \mathbb{R}$ ("$U \approx u$")

Obs. 2:

$$x_i + z_j = (i+j)h \in \mathcal{G}_h$$

$$\Longrightarrow L_h : \underset{\substack{\uparrow \\ \text{functions} \\ \mathcal{G}_h \to \mathbb{R}}}{F(\mathcal{G}_h)} \to F(\mathcal{G}_h) \text{ well-def.}$$

Scheme:

$$(4) \quad U + F(L_h U) = f \quad \text{in } \mathcal{G}_h$$

"FDM – quadrature"

Rem. 1: Not yet implementable:

iii) Non-lin. in U! $\rightsquigarrow$ iterations / Newton meth. to solve ...

i) $\mathcal{G}_h$ unbnd.! $\rightsquigarrow$ restrict to bnd. subset + B.C.

ii) Specify μ_i (later)

D. Properties of the scheme

Consistent:

"$|\text{scheme} - \text{eq'n}| \xrightarrow[h \to 0]{} 0$"

$$|\varphi - F(L_h\varphi) - f - (\varphi - F(L\varphi) - f)|$$

$$\leq L_F |L_h\varphi - L\varphi| \xrightarrow[h\to 0]{(A1)} 0 \quad \forall \varphi \in C_b^\infty$$

L^∞-stable: $\quad (\|u_h\|_\infty \leq K \quad \forall h>0)$

$$\|u_h\|_\infty \leq \|f\|_\infty$$

"Pf.:" Assume $u_h(\bar{x}) = \sup u_h$ (simplification)

Scheme (4) at $\bar{x}$:

$$(\sup u_h =)\, u_h(\bar{x}) = -\overbrace{F(\underbrace{L_h u_h(\bar{x})}_{\leq 0 \text{ by Lem. 2}})}^{\geq 0 \text{ by (F1)}} + f(\bar{x}) \leq \|f\|_\infty$$

Similarly, $\inf u_h \geq -\|f\|_\infty$ $\square$

Monotone:

$$S(x, h, u_h(x), \underbrace{[u_h]}_{\{u_h(x+z_i)\}_{i\neq 0}}) = 0$$

$S(x, h, t, [u])$ monotone (Ba.-Sou.) if

$$[u] \geq [v] \implies S(x_i, h, t, [u]) \leq S(x_i, h, t, [v])$$

component wise

Here: $S(x_i, h, t, [u]) =$

$$= t + F\Big(\tfrac{1}{2} \sum \big([u]_{x_i}(z_i) - 2t + \sum [u]_{x_i}(-z_i)\big) \mu_i\Big) - f(x_i)$$

Hence scheme is monotone

Thm.: ("Barles–Souganidis 1991")

If a scheme is consistent, monotone, and L^∞-stable, and the limit eq'n satisfy (the strong) comp. principle, then num. sol'n conv. loc. unif. to visc. sol'n of limit eq'n.

Cor: $U_h \to u$ loc. unif where u visc. sol'n of (3).

Rem: i) Only L^∞-bnd's on $\{U_h\}$!!

ii) General results:

- Biswas, Jakobsen, Karlsen, SINUM, 2010

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E. Construction of L_h

a) Finite meas. w. comp. supp

$$\mu \rightsquigarrow \boxed{\mu_{r,R} := \mathbb{1}_{r<|z|<R}\,\mu}\ ; \quad L \rightsquigarrow L_{r,R}$$

Obs ? :

i) $\operatorname{supp} \mu_{r,R} \subset \{z : r<|z|<R\}$

ii) $\mu_{r,R}(\mathbb{R}^N) = \int \mathbb{1}_{r<|z|<R}\,\mu \leq \int \left|\frac{z}{r}\right|^2 \mathbb{1}_{r<|z|<R}\,\mu < \infty$

Lem ? :

$$|L_{r,R}\varphi(x) - L\varphi(x)| = \Big|\int \{\varphi \cdots\} \underbrace{(\mathbb{1}_{r<|z|<R} - 1)}_{\substack{\longrightarrow 0 \\ r\to 0,\ R\to\infty}} \mu(dz)\Big| \underset{\substack{r\to 0\\ R\to\infty}}{\longrightarrow} 0$$

dom. conv.

b) Quadrature:

$$\int_{-R}^{R}\psi(z)\,dz \simeq \sum_{i=-P_h}^{P_h} a_i\,\psi(z_i) =: Q_h[\psi] \ , \ h>0$$

coeff.   nodes

Conv.: $\left|\int\psi\,dz - Q_h[\psi]\right| \xrightarrow[h\to 0]{} 0 \quad \forall \psi \in C_b$

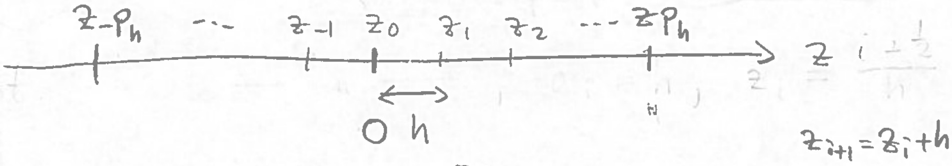
Positive: $a_i \geq 0 \quad \forall i$

Equi. dist. nodes: $z_i - z_{i-1} = h$ $\qquad Q_h$ wh

Comp. stencil: $P_h < \infty$

Inf. conf $\Rightarrow Q_h(K)=K$ ($\Leftarrow$ 1. order?)

Ex. S: "Riemann": Take $P_h \in \mathbb{N}$: $h := \frac{R}{P_h}$, $a_i = h$, $z_i = ih$



$z_{i+1} = z_i + h$

$$\left|\int\psi\,dz - Q_h(\psi)\right| \leq \sum \int_{z_i}^{z_{i+1}} |\psi(z) - \psi(z_i)|\,dz \leq \|D\psi\|_\infty \cdot h$$

first order meth.

$\rightarrow$ conv., pos., equi. di

Rem. ? $\qquad$ Others: Trapezoidal, Simpson, (4)...

High order, pos., equi.di : Midpt. (2),

—— " ——, — : Gauss, GLL

c) Discrete L

$$L_{h,r,R}\varphi(x) := \frac{1}{2} Q_h\left[\{\varphi(x+\cdot) - 2\varphi + \varphi(x-\cdot)\}\mu_{r,R}\right]$$

$$= \frac{1}{2}\sum\{\varphi(x+z_i) - 2\varphi(x) + \varphi(x-z_i)\}\underbrace{\mu_{r,R}(z_i)\,a_i}_{\mu_i \geq 0}$$

Chk.: $L_{h,r,R}\varphi \xrightarrow[R\to\infty]{h,r\to 0} L\varphi \quad \forall \varphi \in C^2 \cap L^\infty$